“Switched Systems II”, FrB11.5

Computation of LQ Control for Continuous-Time Bimodal Switched Linear Systems

Naoyuki Hara and Keiji Konishi

Osaka Prefecture University

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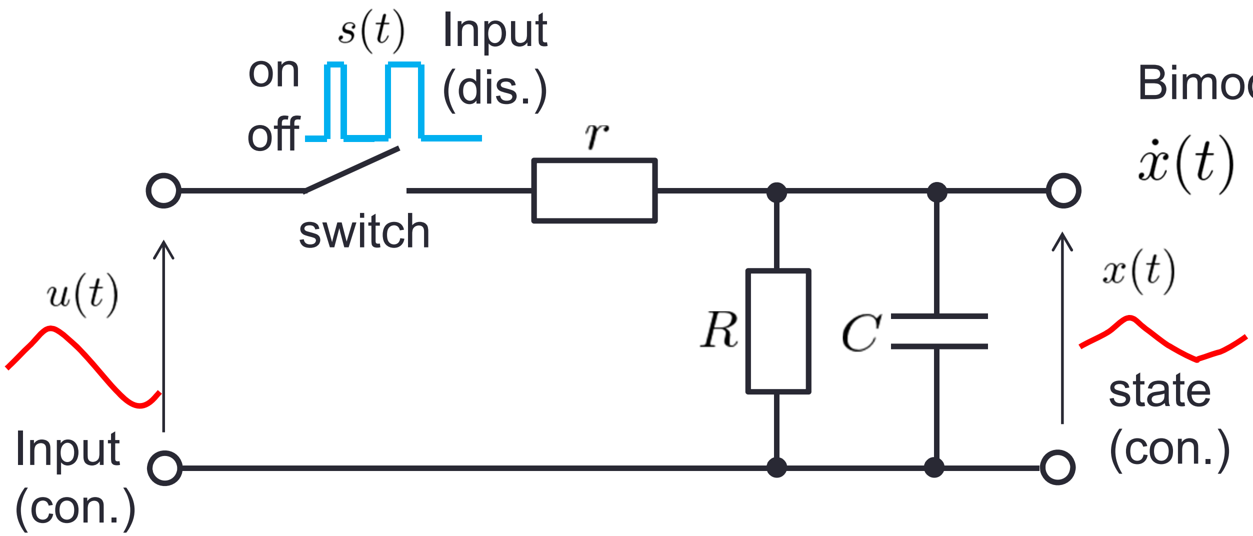
Introduction

- Switched Systems
  - Multiple subsystems
  - Dynamics is determined by an active subsystem

D. Liberzon, *Switching in Systems and Control*, 2003
Z. Sun & S.S. Ge, *Switched Linear Systems: Control and Design*, 2005

- Externally forced switched systems
  - An external signal (control input) determines the active system
  - Two types of input: discrete-valued and continuous-valued inputs



Bimodal switched linear system

$$\dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t)$$

$$A_1 = -\frac{r+R}{rRC} \qquad B_1 = \frac{1}{rC}$$

$$A_2 = -\frac{1}{CR} \qquad B_2 = 0$$

Introduction

- Optimal control
  - One of the research topics for switched systems

    Zhu & Antsaklis (2015), Xu & Antsaklis (2004), Bemporad et al. (2002) Bengea & DeCarlo (2005), Seatzu (2006), Patino et al. (2009)

  - Switched LQ problem for continuous-time linear switched systems

    Riedinger (2014), Riedinger et al. (2015), Das et al. (2008)

    - Computation method based on TPBV problem (no continuous-valued input term) (Das et al. 2008)

- In this paper,
  - Adopt the approach, Das et al. (*Automatica*, 2008)
  - LQ control of bimodal switched linear systems
  - Computation method for continuous- and discrete-valued signals
  - *No assumption* made for switching instants, number, and sequence

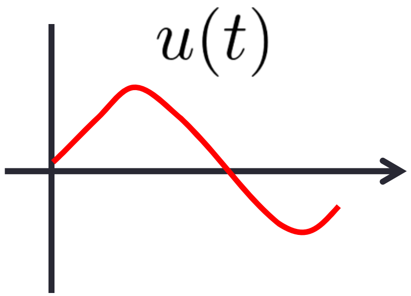
Outline

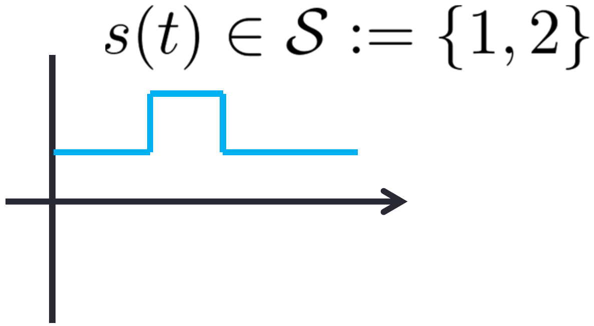
- Problem Formulation and Optimality Condition
- Numerical Computation
- Numerical Example

Problem Formulation

- Bimodal Switched Linear System

$$\dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t), \;\; x(0) = x_0$$





Subsystem 1 (A_1, B_1)

Subsystem 2 (A_2, B_2)

$$\dot{x}(t) = A_2 x(t) + B_2 u(t) + (A_{12}x(t) + B_{12}u(t))\sigma(t)$$

$$(A_{12} := A_1 - A_2, \;\; B_{12} := B_1 - B_2)$$

$$\sigma(t) \in \Sigma_s := \{0, 1\}$$

Problem Formulation

$$\dot{x}(t) = A_2 x(t) + B_2 u(t) + (A_{12} x(t) + B_{12} u(t))\sigma(t)$$

$$\sigma(t) \in \Sigma_s := \{0, 1\} \longrightarrow \sigma(t) \in \bar{\Sigma}_s := [0, 1]$$

Relaxation technique used in optimal control for SS

- Optimal Control Problem

$$J(x_0, u, \sigma) := \frac{1}{2}\int_0^{t_f} x^T(t) Q x(t) + u^T(t) R u(t) dt + \frac{1}{2} x^T(t_f) Q_f x(t_f)$$

$$\mathcal{P} : \min_{u(\cdot), \sigma(\cdot) \in \bar{\Sigma}_s} J$$

s.t. Dynamics eq.

Necessary Conditions (Minimum principle)

$$\dot{x} = \frac{\partial}{\partial \lambda} H$$

$$\dot{\lambda} = -\frac{\partial}{\partial x} H$$

$$x(0) = x_0, \lambda(t_f) = Q_f x(t_f)$$

$$H(x, \lambda, u, \sigma) = \min_{\bar{u}(\cdot), \bar{\sigma}(\cdot)} H(x, \lambda, \bar{u}, \bar{\sigma})$$

H : Hamiltonian

λ : Adjoint variable

Optimality Condition

Theorem 1 (necessary condition for optimality)

$$a(\lambda) := \lambda^T B_{12} R^{-1} B_{12}^T \lambda$$
$$b(x,\lambda) := \lambda^T (A_{12}x - B_{12} R^{-1} B_2^T \lambda)$$

Assumed to be not simultaneously zero (⇒relaxed switching signal has a bang-bang type solution)

Differential equation with boundary value (Two-Point Boundary Value Problem)

$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = S(\sigma) \begin{bmatrix} \lambda \\ x \end{bmatrix}, \ x(0) = x_0, \ \lambda(t_f) = Q_f x(t_f),$$

$$S(\sigma) := \begin{bmatrix} -(A_2 + A_{12}\sigma)^T & -Q \\ -(B_2 + B_{12}\sigma) R^{-1} (B_2 + B_{12}\sigma)^T & (A_2 + A_{12}\sigma) \end{bmatrix}$$

Switching Condition (Discrete-valued Input)

$$\sigma(t) = \begin{cases} 0 \ \text{ if } \ b(x,\lambda) > \frac{1}{2} a(\lambda) \\ 1 \ \text{ if } \ b(x,\lambda) < \frac{1}{2} a(\lambda) \\ \text{either } 0 \text{ or } 1 \text{ if } b(x,\lambda) = \frac{1}{2} a(\lambda) \end{cases}$$

Continuous-valued input

$$u(t) = -R^{-1} (B_2 + B_{12}\sigma)^T \lambda,$$

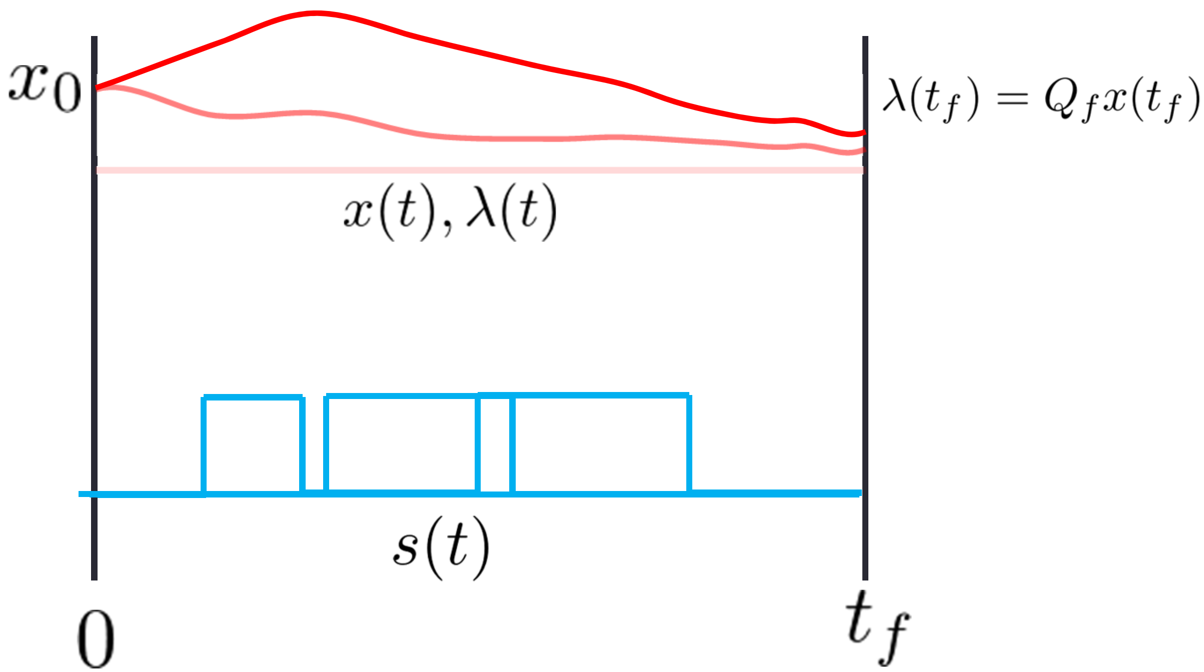
Numerical Computation

- Two-Point Boundary Value Problem

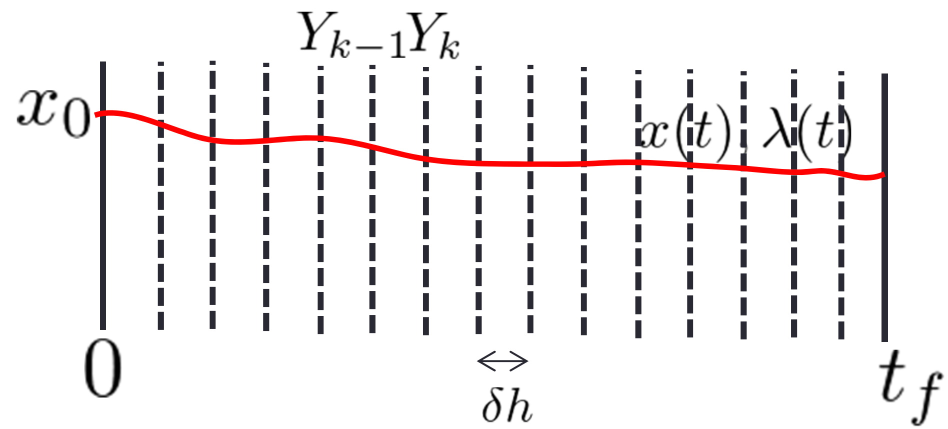
$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = S(\sigma) \begin{bmatrix} \lambda \\ x \end{bmatrix}, \ x(0) = x_0, \ \lambda(t_f) = Q_f x(t_f),$$

$$S(\sigma) := \begin{bmatrix} -(A_2 + A_{12}\sigma)^T & -Q \\ -(B_2 + B_{12}\sigma)R^{-1}(B_2 + B_{12}\sigma)^T & (A_2 + A_{12}\sigma) \end{bmatrix}$$

- No assumptions on switching:
  - time
  - number
  - sequence
- Use relaxation method
- Starting from a guess solution, update the estimated solutions.



Numerical Computation



- Discretization
  - Grid size δh
- Estimate at k-th grid point

$$Y_k = \begin{bmatrix} x(k \cdot \delta h) \\ \lambda(k \cdot \delta h) \end{bmatrix}$$

$$\sigma_k = \sigma(k \cdot \delta h)$$

$$\sigma(t) = \begin{cases} 0 \ \text{ if } \ b(x,\lambda) > \frac{1}{2}a(\lambda) \\ 1 \ \text{ if } \ b(x,\lambda) < \frac{1}{2}a(\lambda) \\ \text{either } 0 \text{ or } 1 \text{ if } b(x,\lambda) = \frac{1}{2}a(\lambda) \end{cases}$$

- Dynamics eq.

$$Y_k = e^{S(\sigma_{k-1})\delta h} Y_{k-1}$$ should be satisfied

$$E_k = Y_k - e^{S(\sigma_{k-1})\delta h} Y_{k-1} \ \ (k = 1, 2, \ldots, M)$$

- Find the corrections such that

$$(Y_k + \Delta Y_k) - e^{S(\sigma_{k-1})\delta h}(Y_{k-1} + \Delta Y_{k-1}) = 0$$

Numerical Computation

$$x(0) = x_0$$
$$\lambda(t_f) = Q_f x(t_f)$$
$$(Y_k + \Delta Y_k) - e^{S(\sigma_{k-1})\delta h}(Y_{k-1} + \Delta Y_{k-1}) = 0$$

System of linear equations

$$G \begin{bmatrix} \Delta Y_0 \\ \Delta Y_1 \\ \Delta Y_2 \\ \vdots \\ \Delta Y_M \end{bmatrix} = (-G) \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_M \end{bmatrix} + \begin{bmatrix} x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$G(\sigma_0, \sigma_1, \sigma_2, \cdots, \sigma_{M-1})$$

$$G := \begin{bmatrix} g_0 & & & & & \\ g_{1,0} & g_2 & & & & \\ & g_{1,1} & g_2 & & & \\ & & & \ddots & & \\ & & & & g_{1,M-1} & g_2 \\ & & & & & g_3 \end{bmatrix}$$

$$g_0 = [0_n, I_n], \; g_{1,k} = -e^{S(\sigma_k)\delta h},$$
$$g_2 = I_{2n}, \; g_3 = [I_n, -Q_f]$$

Update the estimate

$$Y_k \leftarrow Y_k + \alpha \Delta Y_k$$
$$0 < \alpha < 1$$

Compute the mode sequence with newly obtained Y_k and repeat the same procedure until corrections become sufficiently small.

Numerical Example 1

- Bimodal switched linear system

$$\dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t),\ \ x(0) = x_0$$

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ stable

$$A_2 = \begin{bmatrix} -3 & 3 \\ 2 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$ unstable

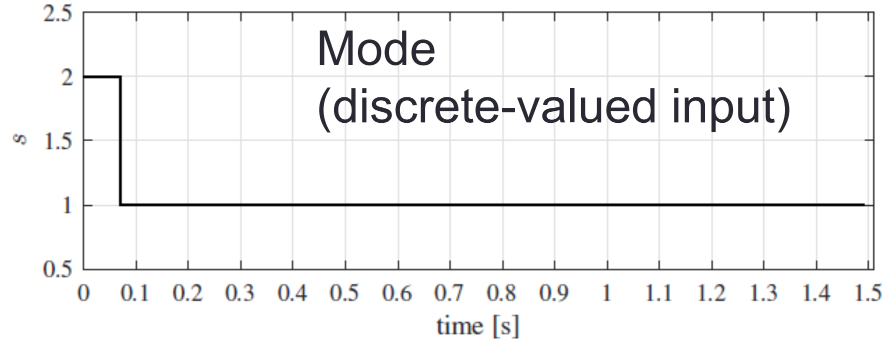
- Parameters

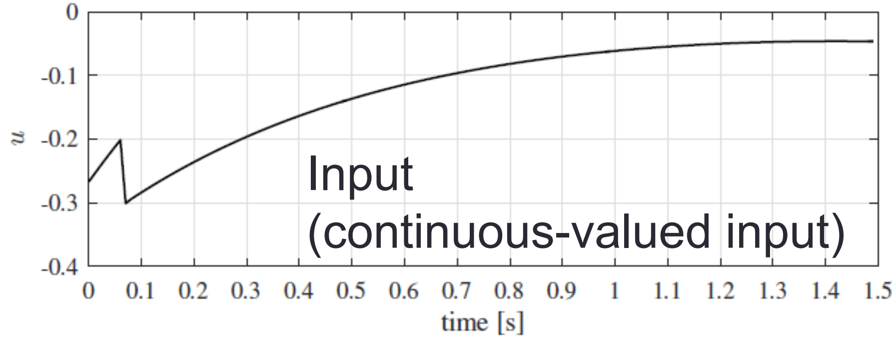
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 1 \quad Q_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad t_f = 1.5$$

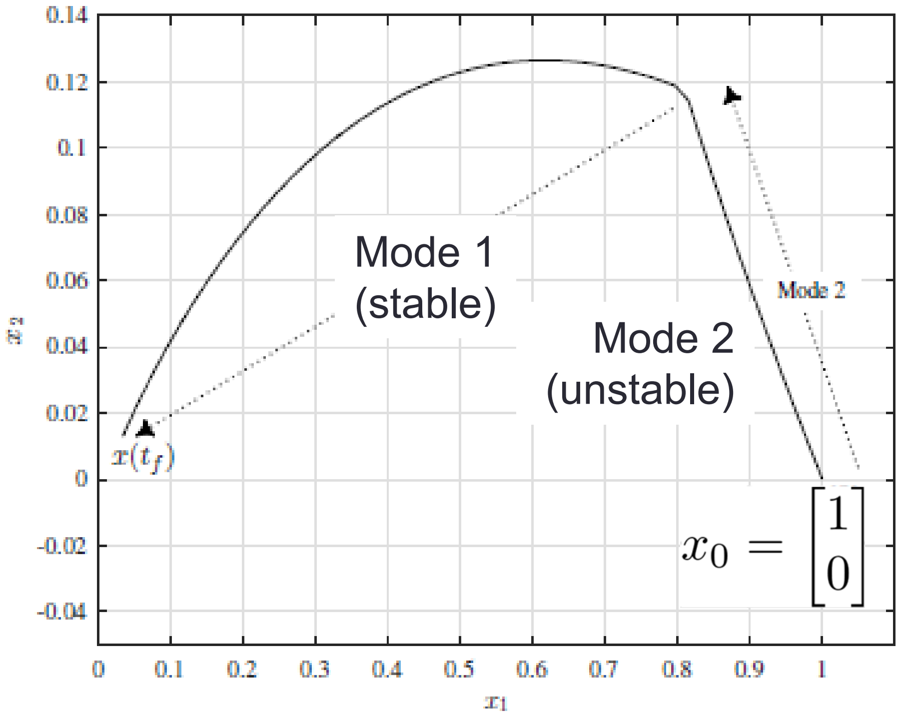
$$\frac{1}{2}\int_0^{t_f} x^T(t)Qx(t) + u^T(t)Ru(t)dt + \frac{1}{2}x^T(t_f)Q_f x(t_f)$$

$$\delta h = 0.01$$

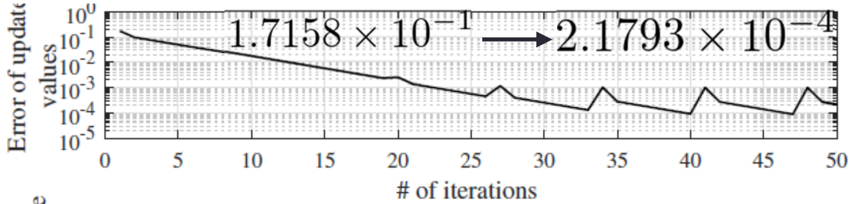
$$\alpha = 0.2$$



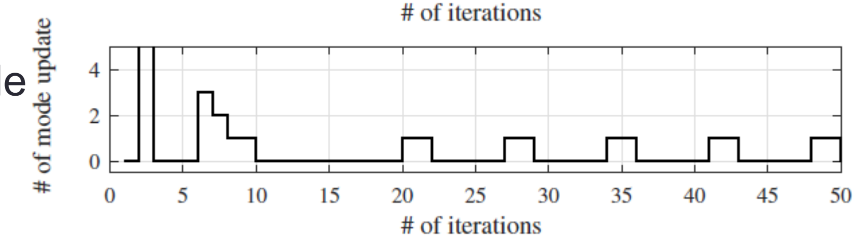




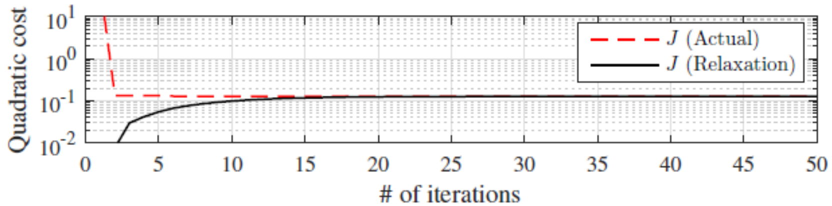
Corrections



Number of mode update



Quadratic cost



- Error reduces exponentially until 25 iterations
- Number of mode update happens relatively frequently up to 10 iterations
- Quadratic Cost:
  0.1273

Numerical Example 2

- Bimodal Switched Linear System (Example 2, Xu & Antsaklis, *IEEE TAC*, 2004)

$$\dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t),\ \ x(0) = x_0$$

$$A_1 = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

anti-stable

(eigenvalues: 1, 6)

$$A_2 = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

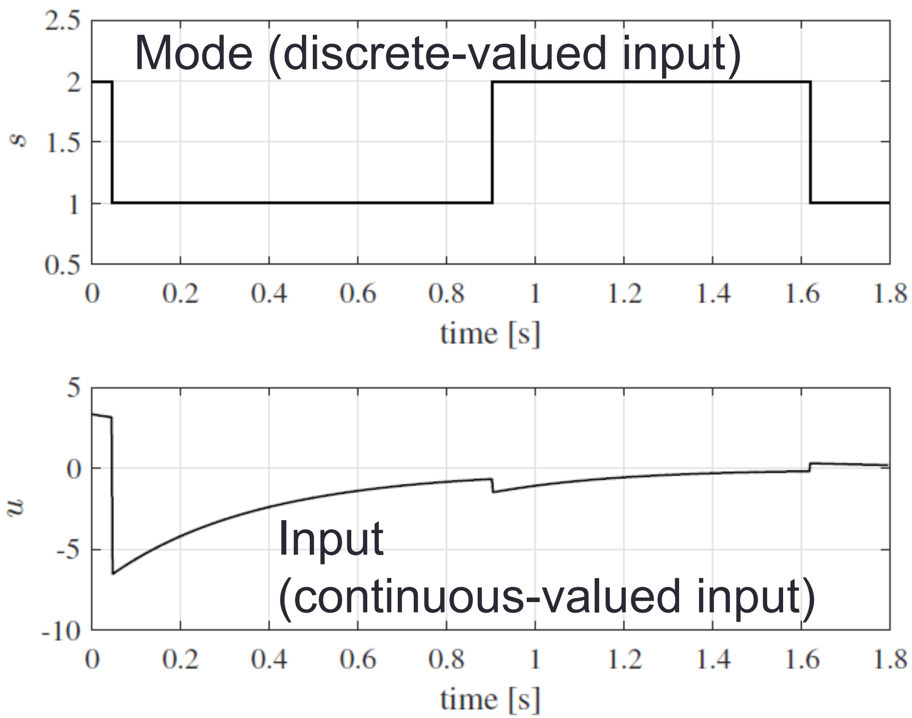
anti-stable

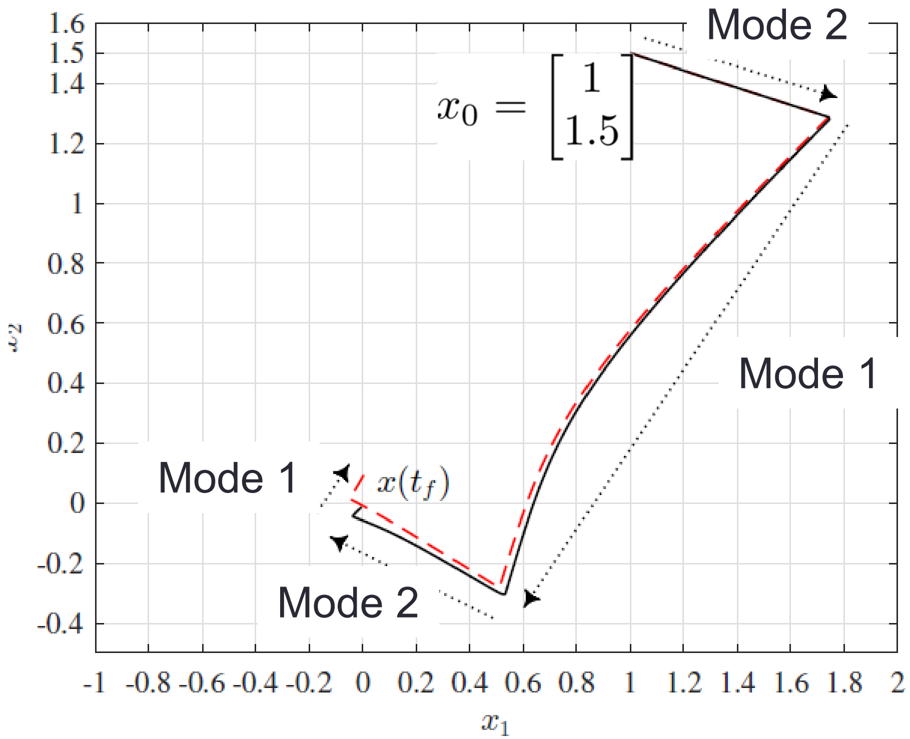
- Parameters

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ \ R = 1\ \ Q_f = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\ \ t_f = 1.8$$

$$\delta h = 0.002 \quad \frac{1}{2}\int_0^{t_f} x^T(t)Qx(t) + u^T(t)Ru(t)dt + \frac{1}{2}x^T(t_f)Q_f x(t_f)$$

$$\alpha = 0.005$$

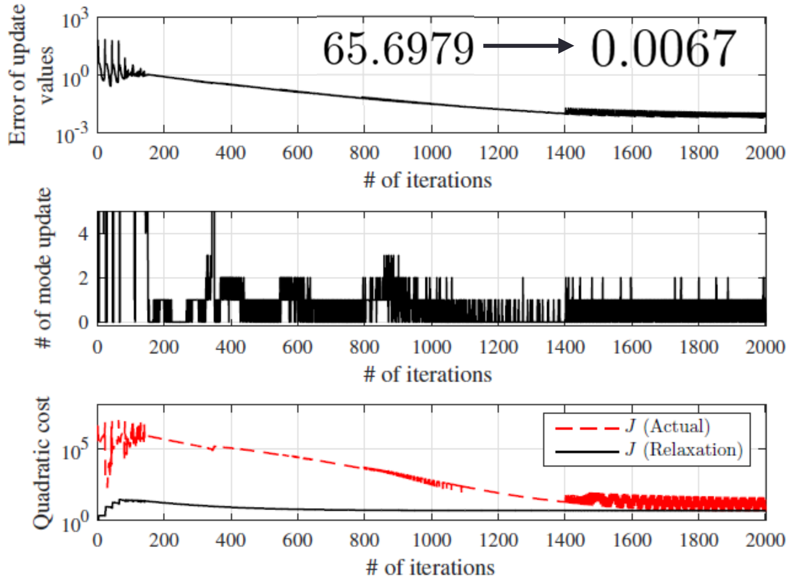




Corrections

Number of mode update

Quadratic cost



- Two modes are alternately used.
- Require more iterations than Example 1
- Quadratic Cost: 4.8307

Summary

- LQ control for bimodal switched linear systems
- Continuous- and discrete-valued inputs
- No assumptions made for switching
- TPBV problem is solved based on the relaxation method

Future work

a. Extension to multi-mode system (more than two)
b. Convergence analysis / parameter choice
c. Electrical circuit experiment

Thank you!

Questions & Comments: n-hara@eis.osakafu-u.ac.jp