Primal Formulation of Parallel Model Predictive Control

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Introduction

- Model Predictive Control (MPC)
  - Uses real-time optimization, requires a greater computational effort
  - This can be a drawback when applied to fast systems or available computational capabilities are not sufficient.

- Two approaches to overcome the drawback
  - Explicit MPC: replace on-line optimization with an explicit function

    A. Bemporad, et al., *Automatica*, 2002

    ➢ For relatively small systems
  - Implicit MPC: speed up on-line optimization

    e.g. Y. Wang & S. Boyd, *IEEE Contr. Syst. Tech.*, 2010

    ➢ For large systems

Introduction

- Parallel computation capabilities are becoming common
  - Advent of many-core CPUs and GPUs
  - Parallelizable algorithms should be designed.
- Use of parallel computation in MPC
  - Parallel move-blocking
    S. Long, E.C. Kerrigan, K.V. Ling, and G.A. Constantinides, *CDC & ECC*, 2011
  - Tailored parallel optimization algorithms
    S.D. Cairano, M. Brand, and S.A. Bortoff, *Int. J Control*, 2013, I. Nielsen and D. Axehill, *CDC* 2015,
    L. Ferranti and T. Keviczky, *CDC* 2015,
    N. Hara, Y. Kagitani, and K. Konishi, *ICCAS,* 2014, N. Hara and K. Konishi, CDC 2016
- N. Hara, et al. (ICCAS 2014), N. Hara and K. Konishi (CDC 2016)
  - An optimization problem is decomposed to several sub-problems.
  - The sub-problems are formulated based on the dual of the QP.
  - Problems and Questions: size tends to be larger, clear interpretation for the sub-problems

- In this paper,
  - We provide a primal formulation-based Parallel MPC.

MPC formulation

Discrete-time LTI system with constraints

$$x(t+1) = Ax(t) + Bu(t),\ \ t = 0, 1, 2,$$
$$F_u u(t) \le c_u,\ \ F_x x(t) \le c_x$$

$$x(t) \in \mathbb{R}^n \quad u(t) \in \mathbb{R}^m \quad F_u, F_x, c_u, c_x$$

Finite Horizon Optimal Control Problem

$$\min_{U} \sum_{k=0}^{N_p-1} \{x_k^T Q x_k + u_k^T R u_k\} + x_{N_p}^T Q_f x_{N_p}$$

$$N_p \ge 1$$
$$Q, R, Q_f > 0$$

$$\begin{aligned} \text{s.t.}\quad & x_0 = x(t), \\ & F_u u_k \le c_u,\ \ k = 0, 1, \ldots, N_p - 1, \\ & F_x x_k \le c_x,\ \ k = 1, 2, \ldots, N_p - 1, \\ & F_{x_f} x_{N_p} \le c_{x_f} \end{aligned}$$

$$U := \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \vdots \\ u_{N_p-1} \\ x_{N_p} \end{bmatrix}$$

MPC formulation

Formulated as a QP (Quadratic Programming) Problem

$$\mathcal{P}_{QP} \quad \min_{U} \ \frac{1}{2} U^T H_{PP} U$$
$$\text{s.t.} \quad VU \leq W$$
$$V_{eq} U = W_{eq} x_0, \ \ x_0 = x(t),$$

$H_{PP}, V, W, V_{eq}, W_{eq}$
appropriate matrices

MPC

- We need to solve the QP-problem, $\mathcal{P}_{QP}$, in real-time.
- This motivates us to reduce the computational time.

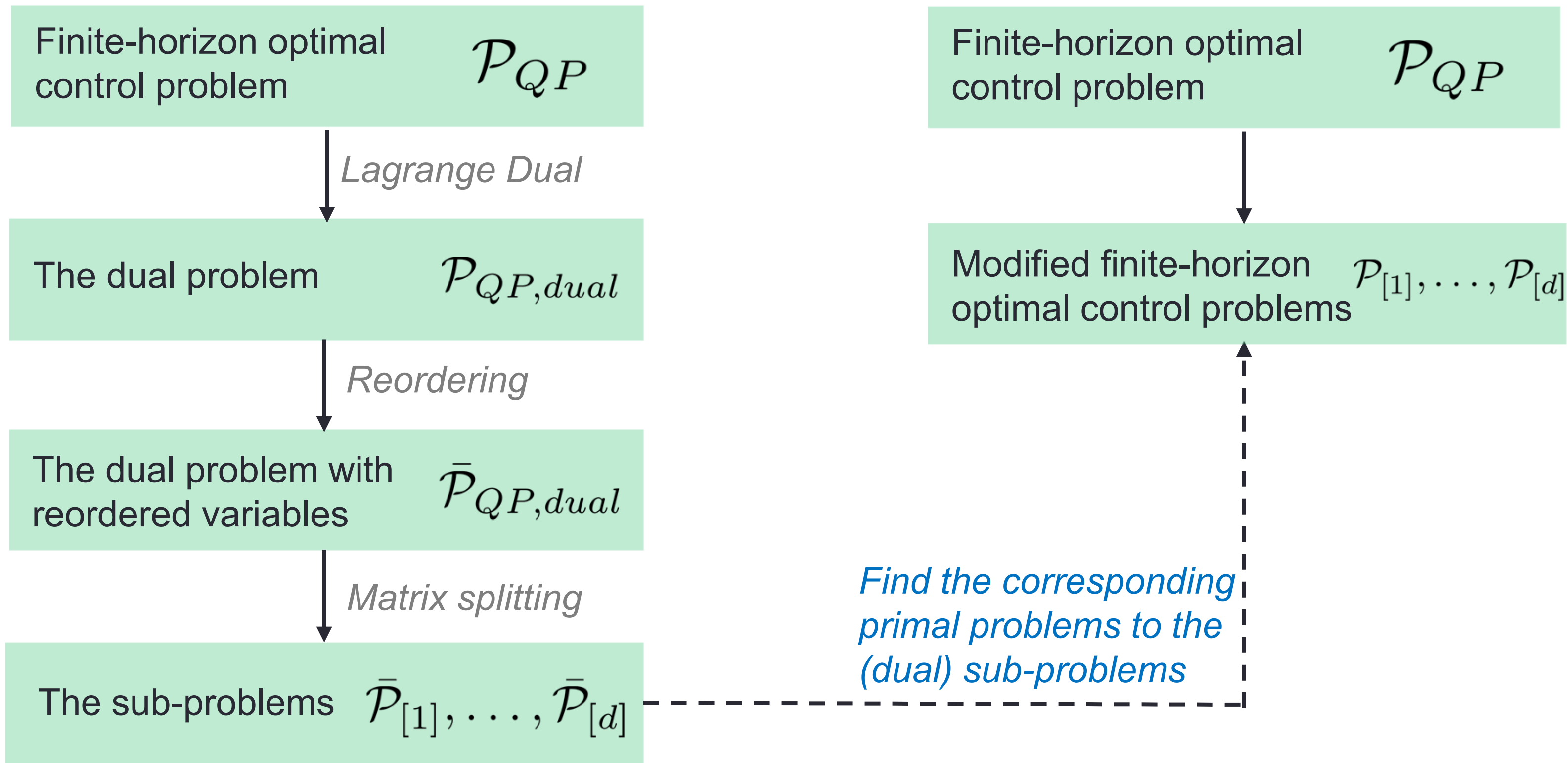
Parallelization based on matrix splitting technique (N. Hara & K. Konishi, CDC 2016)

- Decomposition of $\mathcal{P}_{QP}$ into the parallelizable sub-problems.
- The sub-problems can be solved in parallel.

Dual-Based and Primal-Based Approaches

Previous Approach (Dual-Based Approach)
N. Hara & K. Konishi, CDC 2016

This study: Primal-Based Approach



Problems and Questions

A) The size of the dual variables is larger. → Kept almost the same

B) Any relationship between the sub-problems in terms of control problem? → The sub-problems have the modified forms of the FOCP.

Primal Formulation-Based Parallel MPC

Define some variables

$d > 1$: Parallelization degree; divisor of the prediction horizon N_p

$b_s = \frac{N_p}{d}$: Size of the prediction horizon for each sub-problem

$\mathcal{P}_{[1]}, \mathcal{P}_{[2]}, \ldots, \mathcal{P}_{[d]}$: The sub-problems (optimization problems)

$$U_{[1]}^{(j)} := \begin{bmatrix} u_{[1],0}^{(j)} \\ x_{[1],1}^{(j)} \\ u_{[1],1}^{(j)} \\ x_{[1],2}^{(j)} \\ \vdots \\ u_{[1],b_s-1}^{(j)} \\ x_{[1],b_s}^{(j)} \end{bmatrix} \qquad U_{[i]}^{(j)} := \begin{bmatrix} x_{[i],0}^{(j)} \\ u_{[i],0}^{(j)} \\ x_{[i],1}^{(j)} \\ u_{[i],1}^{(j)} \\ x_{[i],2}^{(j)} \\ \vdots \\ u_{[i],b_s-1}^{(j)} \\ x_{[i],b_s}^{(j)} \end{bmatrix}$$

: Solution of i-th sub-problem $\mathcal{P}_{[i]}$ at j-th iteration.

$U_{[1]}^{(0)}, U_{[i]}^{(0)} (i = 2, 3, \ldots, d)$ are assumed to be given

Primal Formulation-Based Parallel MPC

The first sub-problem $\mathcal{P}_{[1]}$ (at iteration step j)

$$\mathcal{P}_{[1]}: \min_{U_{[1]}} \sum_{k=0}^{b_s-1} \left\{ x_{[1],k}^T Q x_{[1],k} + u_{[1],k}^T R u_{[1],k} \right\} + x_{[1],b_s}^T Q x_{[1],b_s}$$

$$\begin{aligned} \text{s.t. } & x_{[1],k+1} = A x_{[1],k} + B u_{[1],k},\ k = 0, 1, \ldots, b_s - 2 \\ & x_{[1],0} = x_0 (= x(t)) \\ & x_{[1],b_s} = A x_{[1],b_s-1} + B u_{[1],b_s-1} - x_{[2],0}^{(j)} \\ & F_u u_{[1],k} \leq c_u,\ \ k = 0, 1, 2, \ldots, b_s - 1, \\ & F_x x_{[1],k} \leq c_x,\ \ k = 1, 2, \ldots, b_s - 1. \end{aligned}$$

- Shorter prediction horizon
- No constraint for the last state prediction
- Perturbation term from the adjacent sub-problem

Primal Formulation-Based Parallel MPC

The sub-problem $\mathcal{P}_{[i]}$ $(i = 2, 3, \ldots, d-1)$ (at iteration step j)

$$\mathcal{P}_{[i]} : \min_{U_{[i]}} \sum_{k=0}^{b_s - 1} \left\{ x_{[i],k}^T Q x_{[i],k} + u_{[i],k}^T R u_{[i],k} \right\} + x_{[i],b_s}^T Q x_{[i],b_s}$$

$$\begin{aligned}
\text{s.t. } x_{[i],1} &= A x_{[i],0} + B u_{[i],0} + A x_{[i-1],b_s}^{(j)}, \\
x_{[i],k+1} &= A x_{[i],k} + B u_{[i],k}, \; k = 1, 2, \ldots, b_s - 2 \\
x_{[i],b_s} &= A x_{[i],b_s-1} + B u_{[i],b_s-1} - x_{[i+1],0}^{(j)} \\
F_u u_{[i],k} &\le c_u, \;\; k = 0, 1, 2, \ldots, b_s - 1, \\
F_x x_{[i],0} &\le c_x - F_x x_{[i-1],b_s}^{(j)}, \\
F_x x_{[i],k} &\le c_x, \;\; k = 1, 2, \ldots, b_s - 1.
\end{aligned}$$

- The initial state is also an optimization problem
- Constraint for the state at the first time step exists

Primal Formulation-Based Parallel MPC

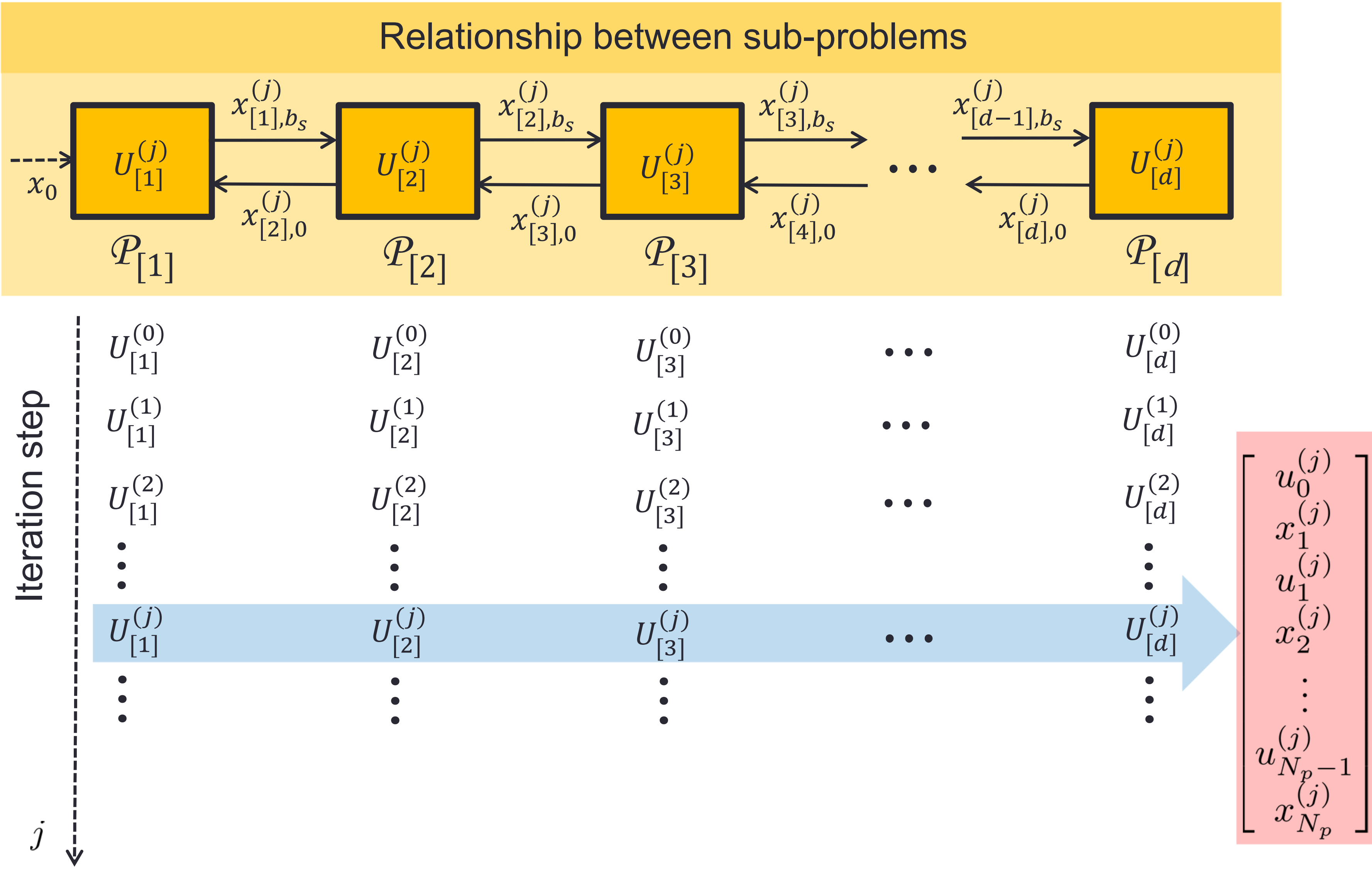
The last sub-problem $\mathcal{P}_{[d]}$ (at iteration step j)

$$\mathcal{P}_{[d]} \ : \ \min_{U_{[d]}} \sum_{k=0}^{b_s-1} \left\{ x_{[d],k}^T Q x_{[d],k} + u_{[d],k}^T R u_{[d],k} \right\} + x_{[d],b_s}^T Q_f x_{[d],b_s}$$

$$\begin{aligned}
\text{s.t. } x_{[d],1} &= A x_{[d],0} + B u_{[d],0} + A x_{[d-1],b_s}^{(j)}, \\
x_{[d],k+1} &= A x_{[d],k} + B u_{[d],k}, \ k = 1, 2, \ldots, b_s - 1 \\
F_u u_{[d],k} &\leq c_u, \ \ k = 0, 1, 2, \ldots, b_s - 1, \\
F_x x_{[d],0} &\leq c_x - F_x x_{[d-1],b_s}^{(j)}, \\
F_x x_{[d],k} &\leq c_x, \ \ k = 1, 2, \ldots, b_s - 1, \\
F_{x_f} x_{[d],b_s} &\leq c_{x_f}.
\end{aligned}$$

- Terminal weight matrix, terminal state constraint
- No perturbation at the end of the state prediction

Primal Formulation-Based Parallel MPC



$$U^{(j)} = \begin{bmatrix} u_0^{(j)} \\ x_1^{(j)} \\ u_1^{(j)} \\ x_2^{(j)} \\ \vdots \\ u_{b_s-1}^{(j)} \\ x_{b_s}^{(j)} \\ u_{b_s}^{(j)} \\ x_{b_s+1}^{(j)} \\ \vdots \\ \vdots \\ u_{N_p-1}^{(j)} \\ x_{N_p}^{(j)} \end{bmatrix} =$$

$$\begin{bmatrix} u_{[1],0}^{(j)} \\ x_{[1],1}^{(j)} \\ u_{[1],1}^{(j)} \\ x_{[1],2}^{(j)} \\ \vdots \\ u_{[1],b_s-1}^{(j)} \\ x_{[1],b_s}^{(j)} \end{bmatrix} = U_{[1]}^{(j)}$$

$$+ \begin{bmatrix} x_{[2],0}^{(j)} \\ u_{[2],0}^{(j)} \\ x_{[2],1}^{(j)} \\ u_{[2],1}^{(j)} \\ x_{[2],2}^{(j)} \\ \vdots \\ u_{[2],b_s-1}^{(j)} \\ x_{[2],b_s}^{(j)} \end{bmatrix} = U_{[2]}^{(j)}$$

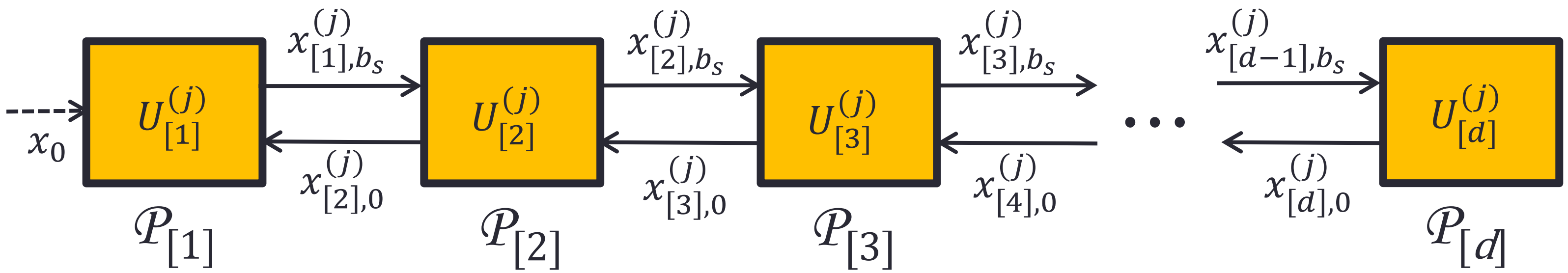
$$+ \cdots$$

$$+ \begin{bmatrix} x_{[d],0}^{(j)} \\ u_{[d],0}^{(j)} \\ x_{[d],1}^{(j)} \\ u_{[d],1}^{(j)} \\ x_{[d],2}^{(j)} \\ \vdots \\ u_{[d],b_s-1}^{(j)} \\ x_{[d],b_s}^{(j)} \end{bmatrix} = U_{[d]}^{(j)}$$

Primal Formulation-Based Parallel MPC

Remarks on the proposed method

1. The original problem $\mathcal{P}_{QP}$ is feasible $\longrightarrow$ sub-problems are feasible.
2. An input sequence obtained at any iteration steps satisfies the constraint. (the state sequence corresponding to the first sub-problem satisfies the constraint)
3. The QP problem is decomposed into d sub-problems along the prediction horizon.
4. Both sparse (as presented here) and dense formulations are possible.
5. The size of each sub-problem can be different each other (for simplicity, we assumed the same size here)
6. The convergence results (CDC 2016) cannot be directly applied to the primal formulation case because weakly regular splitting is employed in this study.



Illustrative Example

Plant

$$x(t+1) = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix} u(t)$$

$$x(0) = \begin{bmatrix} 1.0 \\ 1.3 \end{bmatrix}$$

$$-1 \le u(t) \le +1$$

$$\begin{bmatrix} -0.6 \\ -0.6 \end{bmatrix} \le x(t) \le \begin{bmatrix} +2 \\ +2 \end{bmatrix}$$

MPC design parameters

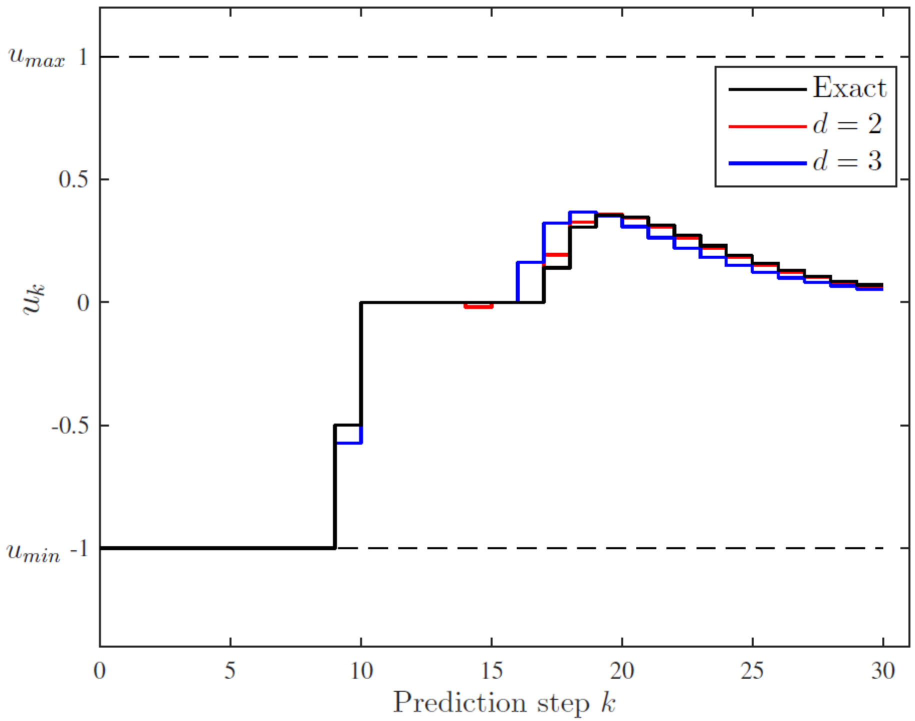
$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad R = 0.1 \quad$ Q_f : solution of the Riccati equation

$$N_p = 30$$

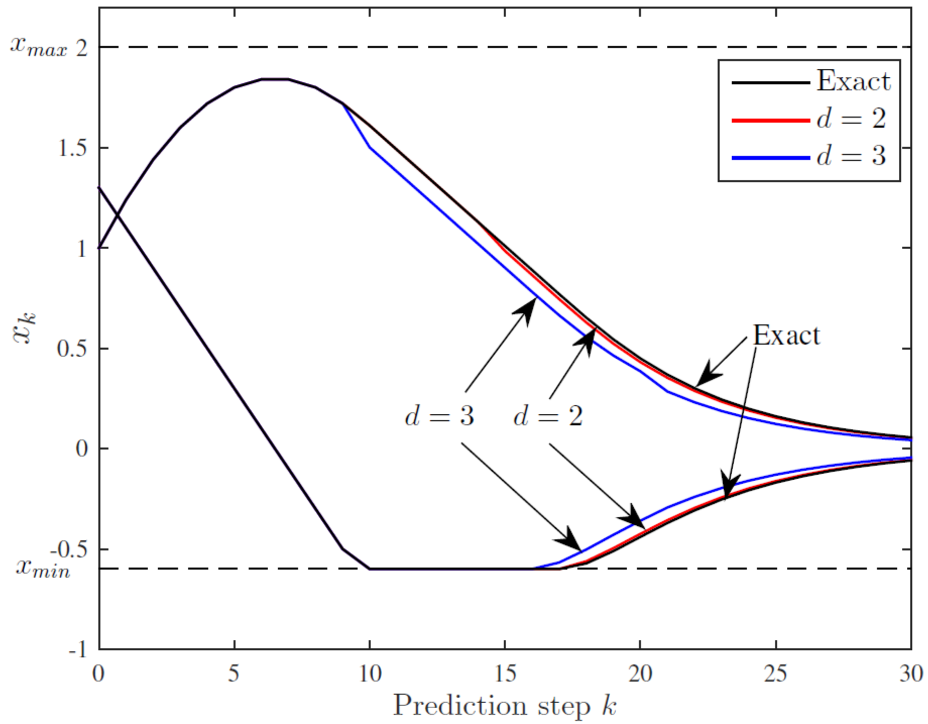
For the parallelization degree, d=2 and d=3, we investigate the performance of the proposed method.

Illustrative Example

Input prediction
(at 50 iterations)

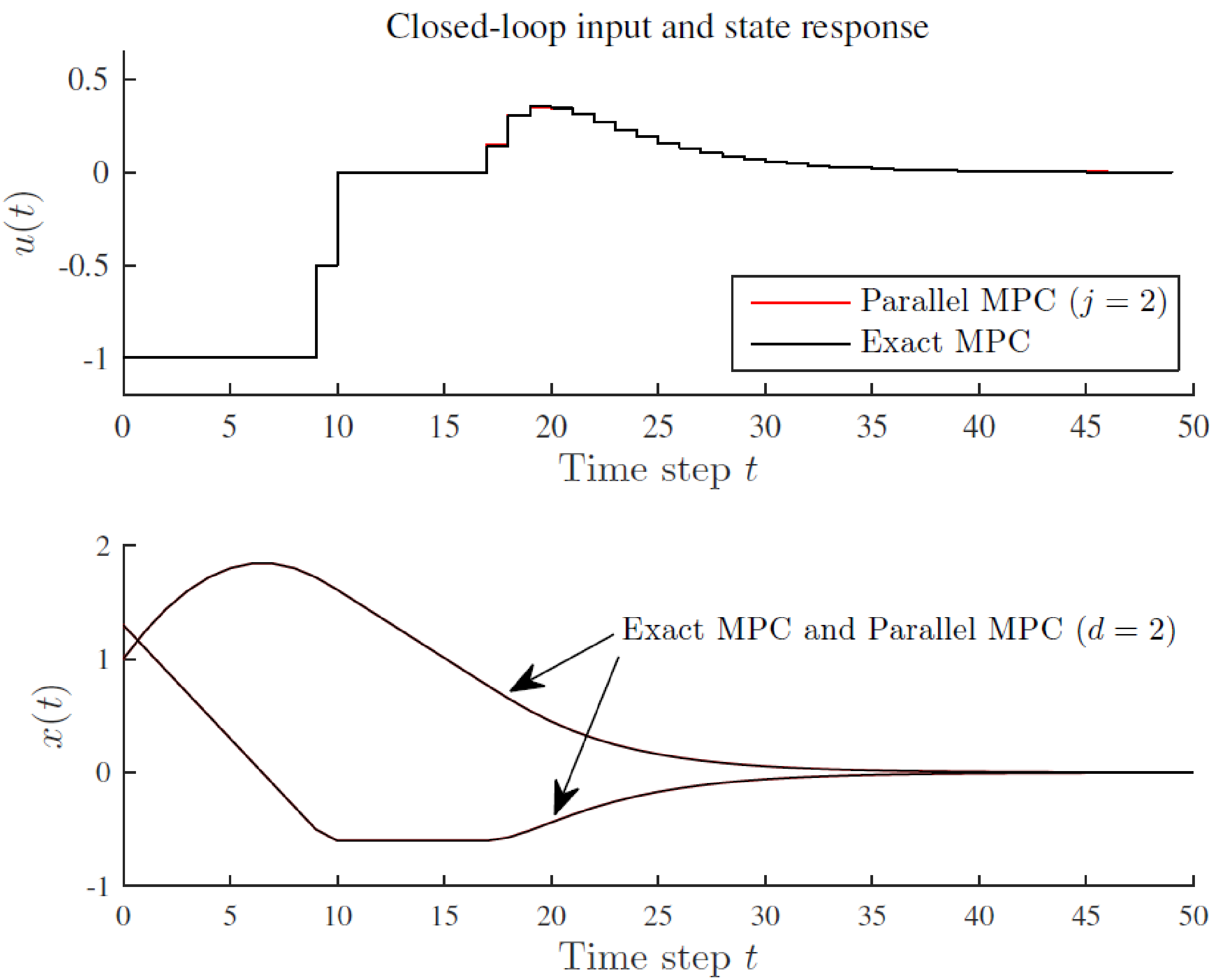


State prediction
(at 50 iterations)



Illustrative Example

Closed-loop response



50 iterations, no warm-starting

Concluding Remarks

- A method for decomposing the QP problem into several parallelizable sub-problems
- The *primal formulation-based* rather than dual-based.
- The constraint satisfaction property of intermediate solutions will be presented at ASCC 2017, December.

- Future research topics
  a. Actual parallel implementation and evaluate the communication overhead of parallelization
  b. Convergence property should be investigated
  c. Some more investigation on the constraint satisfaction

Questions and Comments:
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