WeD6.1

On the Use of Intermediate Solutions in Parallel Model Predictive Control Based on Matrix Splitting

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Introduction

- Model Predictive Control (MPC)
  - Real-time optimization, ability to deal with *constraints*; MPC requires a greater computational effort.
  - This can be a drawback when applied to fast systems or available computational capabilities are not sufficient: explicit MPC.

- Parallel computation capabilities are becoming common
  - Advent of many-core CPUs and GPUs
  - Parallelizable algorithms should be designed.
- Use of parallel computation in MPC

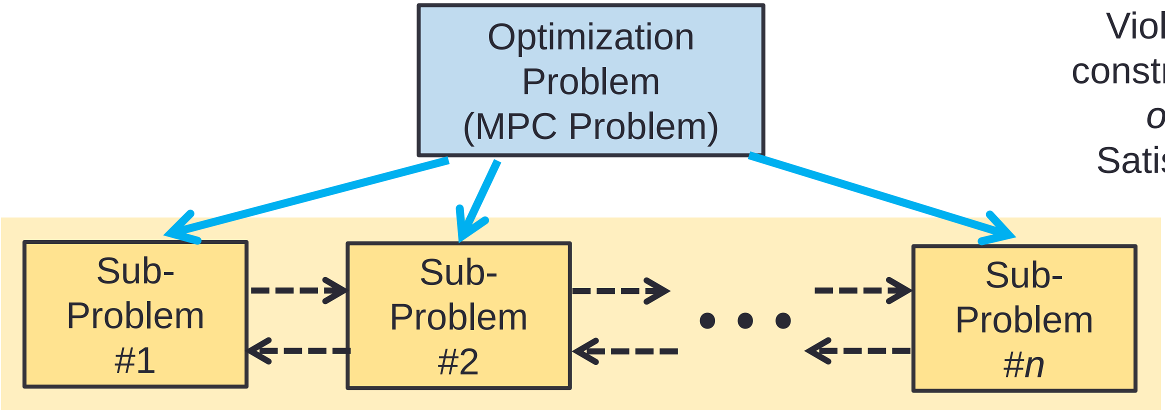
  "Parallel Move-Blocking": S. Long, E.C. Kerrigan, K.V. Ling, and G.A. Constantinides (*CDC & ECC*, 2011)

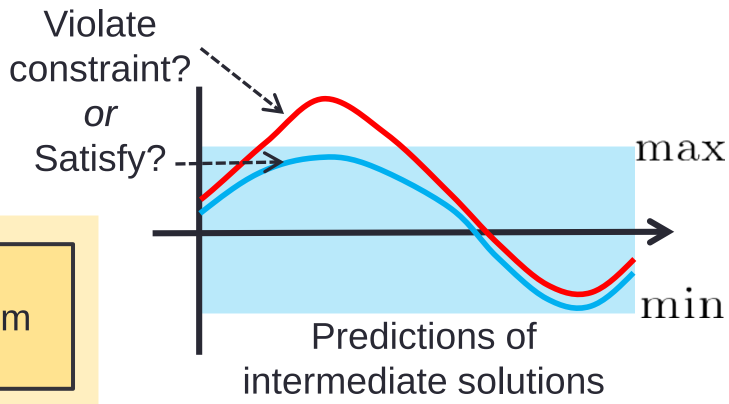
  "Tailored Algorithms": S.D. Cairano, M. Brand, and S.A. Bortoff (*Int. J Control*, 2013), I. Nielsen and D. Axehill (*CDC,* 2015), L. Ferranti and T. Keviczky (*CDC,* 2015), N. Hara, et al. (*ICCAS,* 2014), N. Hara and K. Konishi (*CDC,* 2016)

Introduction

- Our previous work (*CDC* 2016, *ICCAS* 2014)
  - An optimization problem is decomposed into several parallelizable sub-problems by using matrix splitting technique.
  - The sub-problems are solved iteratively to refine the solution; Early termination of iteration gives a good approximate solution suitable for MPC in some situations.

Question: Does an intermediate solution satisfy the constraints?





Contribution of the work:
i) Extension to a system with state and input constraints
ii) Show the constraint satisfaction property (Main Result)

MPC formulation

Discrete-time LTI system with constraints

$$x(t+1) = Ax(t) + Bu(t),\ t = 0, 1, 2, \qquad x(t) \in \mathbb{R}^n$$

$$L_u u(t) + L_x x(t) \le c \qquad u(t) \in \mathbb{R}^m$$

Finite Horizon Optimal Control Problem

$$\min_U \sum_{k=0}^{N_p-1} \{x_k^T Q x_k + u_k^T R u_k\} + x_{N_p}^T Q_f x_{N_p} \qquad N_p \ge 1, \quad Q, R, Q_f > 0$$

$$\text{s.t.}\ x_{k+1} = Ax_k + Bu_k\ (k = 0, 1, \ldots, N_p - 1)$$

$$x_0 = x(t),$$

$$\begin{cases} L_u u_0 \le c, \\ L_u u_k + L_x x_k \le c, k = 1, 2, \ldots, N_p - 1, \\ L_f x_{N_p} \le c_f, \end{cases}$$

$$U := \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \vdots \\ u_{N_p-1} \\ x_{N_p} \end{bmatrix}$$

MPC formulation

Formulated as a QP (Quadratic Programming) Problem

$$\mathcal{P} \quad \min_{U} \ \frac{1}{2} U^T H_{PP} U$$
$$\text{s.t.} \quad VU \leq W$$
$$V_{eq} U = W_{eq} x_0, \ \ x_0 = x(t),$$

$H_{PP}, V, W, V_{eq}, W_{eq}$
appropriate matrices

MPC

- We need to solve the QP-problem, $\mathcal{P}$, in real-time.
- This motivates us to reduce the computational time.

Parallel MPC formulation (N. Hara and K. Konishi, CDC 2016)

- We consider the Lagrange dual and use the matrix splitting.

Dual of QP

$$\min_{z} \ \frac{1}{2} z^T H_{DP} z + q_{DP}^T z \qquad H_{DP} := \begin{bmatrix} V \\ V_{eq} \end{bmatrix} H_{pp}^{-1} \begin{bmatrix} V \\ V_{eq} \end{bmatrix}^T$$

$$\text{s.t.} \ z_a \geq 0, \qquad z = \begin{bmatrix} z_a \\ z_b \end{bmatrix} \begin{matrix} z_a \in \mathbb{R}^{rN_p + \bar{r}} \\ z_b \in \mathbb{R}^{nN_p} \end{matrix} \qquad q_{DP} := \begin{bmatrix} W \\ W_{eq} x_0 \end{bmatrix}$$

$$\Big\downarrow \ s = P_\pi z$$ (Hara & Konishi CDC2016 with modification)

Dual with reordered variables

$$\mathcal{P}_{\text{dual}} : \min_{s} \ \frac{1}{2} s^T H s + q^T s$$

$$\text{s.t.} \ s_i \geq 0, \ (i = 1, 3, 5, \ldots, 2N_p + 1)$$

$$H := \begin{bmatrix} \tilde{S}_1 & \tilde{S}_2 & & & & & & & \cdots & 0 \\ \tilde{S}_2^T & \tilde{S}_3 & \tilde{K}_1 & \tilde{K}_2 & & & & & & \vdots \\ & \tilde{K}_1^T & S_1 & S_2 & & & & & & \\ & \tilde{K}_2^T & S_2^T & S_3 & \tilde{K}_1 & \tilde{K}_2 & & & & \\ & & & \tilde{K}_1^T & S_1 & S_2 & & & & \\ & & & \tilde{K}_2^T & S_2^T & S_3 & & & & \\ & & & & & & \ddots & & & \\ & & & & & \ddots & \tilde{K}_1 & \tilde{K}_2 & & \\ & & & & & \tilde{K}_1^T & S_1 & S_2 & & \\ \vdots & & & & & \tilde{K}_2^T & S_2^T & S_{3f} & -\bar{Q}_f^{-1} L_f^T & \\ 0 & \cdots & & & & & & -L_f \bar{Q}_f^{-1} & L_f \bar{Q}_f^{-1} L_f^T & \end{bmatrix}$$

$$q := \begin{bmatrix} c \\ -Ax_0 \\ c \\ 0 \\ \vdots \\ c \\ 0 \\ c_f \end{bmatrix} \qquad s := \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ \vdots \\ s_{2N_p - 1} \\ s_{2N_p} \\ s_{2N_p + 1} \end{bmatrix}$$

Relationship between U and s : $U = -H_{pp}^{-1} \begin{bmatrix} V \\ V_{eq} \end{bmatrix}^T P_\pi^T s$

Parallel Formulation

$d > 1$: Parallelization degree; divisor of the prediction horizon N_p

$\mathcal{P}_{\mathrm{dual}}$

$$\min_{s} \frac{1}{2} s^T H s + q^T s \quad \text{s.t. } s_i \geq 0, \ i: \text{odd}$$

$\mathcal{P}_{\mathrm{dual,sub}[1]}$

$$s_{[1]}^{(j+1)} = \arg \min_{s_{[1]}} \frac{1}{2} s_{[1]}^T H_i s_{[1]} + q_{[1]}^{(j)T} s_{[1]} \quad \text{s.t. } s_{[1],\ell} \geq 0, \ \ell: \text{odd}$$

Information exchange

$\mathcal{P}_{\mathrm{dual,sub}[2]}$

$$s_{[2]}^{(j+1)} = \arg \min_{s_{[2]}} \frac{1}{2} s_{[2]}^T H_i s_{[2]} + q_{[2]}^{(j)T} s_{[2]} \quad \text{s.t. } s_{[2],\ell} \geq 0, \ \ell: \text{odd}$$

Information exchange

⋮

Information exchange

$\mathcal{P}_{\mathrm{dual,sub}[d]}$

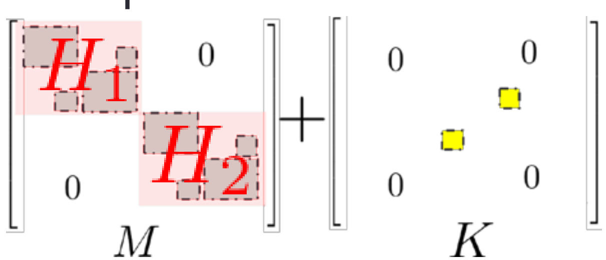
$$s_{[d]}^{(j+1)} = \arg \min_{s_{[d]}} \frac{1}{2} s_{[d]}^T H_i s_{[d]} + q_{[d]}^{(j)T} s_{[d]} \quad \text{s.t. } s_{[d],\ell} \geq 0, \ \ell: \text{odd}$$

Matrix splitting:

$$H = M + K$$

M: Block-diagonal part of *H* wrt *d*

Example: *d=2*



Iteration Step

$0 \quad 1 \quad 2 \quad \cdots \quad j$

$\mathcal{P}_{\mathrm{dual,sub}[1]}$

$$s_{[1]}^{(j+1)} = \arg \min_{s_{[1]}} \frac{1}{2} s_{[1]}^T H_i s_{[1]} + q_{[1]}^{(j)T} s_{[1]}$$

s.t. $s_{[1],\ell} \geq 0, \ \ell$: odd

$s_{[1]}^{(0)} \quad s_{[1]}^{(1)} \quad s_{[1]}^{(2)} \quad \cdots \quad s_{[1]}^{(j)}$

Information exchange

$\mathcal{P}_{\mathrm{dual,sub}[2]}$

$$s_{[2]}^{(j+1)} = \arg \min_{s_{[2]}} \frac{1}{2} s_{[2]}^T H_i s_{[2]} + q_{[2]}^{(j)T} s_{[2]}$$

s.t. $s_{[2],\ell} \geq 0, \ \ell$: odd

$s_{[2]}^{(0)} \quad s_{[2]}^{(1)} \quad s_{[2]}^{(2)} \quad \cdots \quad s_{[2]}^{(j)}$

Information exchange

⋮

Information exchange

$\mathcal{P}_{\mathrm{dual,sub}[d]}$

$$s_{[d]}^{(j+1)} = \arg \min_{s_{[d]}} \frac{1}{2} s_{[d]}^T H_i s_{[d]} + q_{[d]}^{(j)T} s_{[d]}$$

s.t. $s_{[d],\ell} \geq 0, \ \ell$: odd

$s_{[d]}^{(0)} \quad s_{[d]}^{(1)} \quad s_{[d]}^{(2)} \quad \cdots \quad s_{[d]}^{(j)}$

$s^{(j)} = \begin{bmatrix} s_{[1]}^{(j)} \\ s_{[2]}^{(j)} \\ \vdots \\ s_{[d]}^{(j)} \end{bmatrix}$

Input and state sequences satisfy the constraint?

$$U = -H_{pp}^{-1} \begin{bmatrix} V \\ V_{eq} \end{bmatrix}^T P_\pi^T s^{(j)}$$

Main Result

Assumptions

1. The input and state constraints are separable, e.g., lower and upper bound constraints.
2. The original MPC problem is feasible for the initial state x_0.

Theorem 1

Let $s^{(0)}$ be an arbitral initial estimate. Then,

$$\begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \vdots \\ u_{N_p-1} \\ x_{N_p} \end{bmatrix} = -H_{pp}^{-1} \begin{bmatrix} V \\ V_{eq} \end{bmatrix}^T P_\pi^T s^{(j)}$$

i) The input sequence **satisfies the constraint over the entire horizon.**

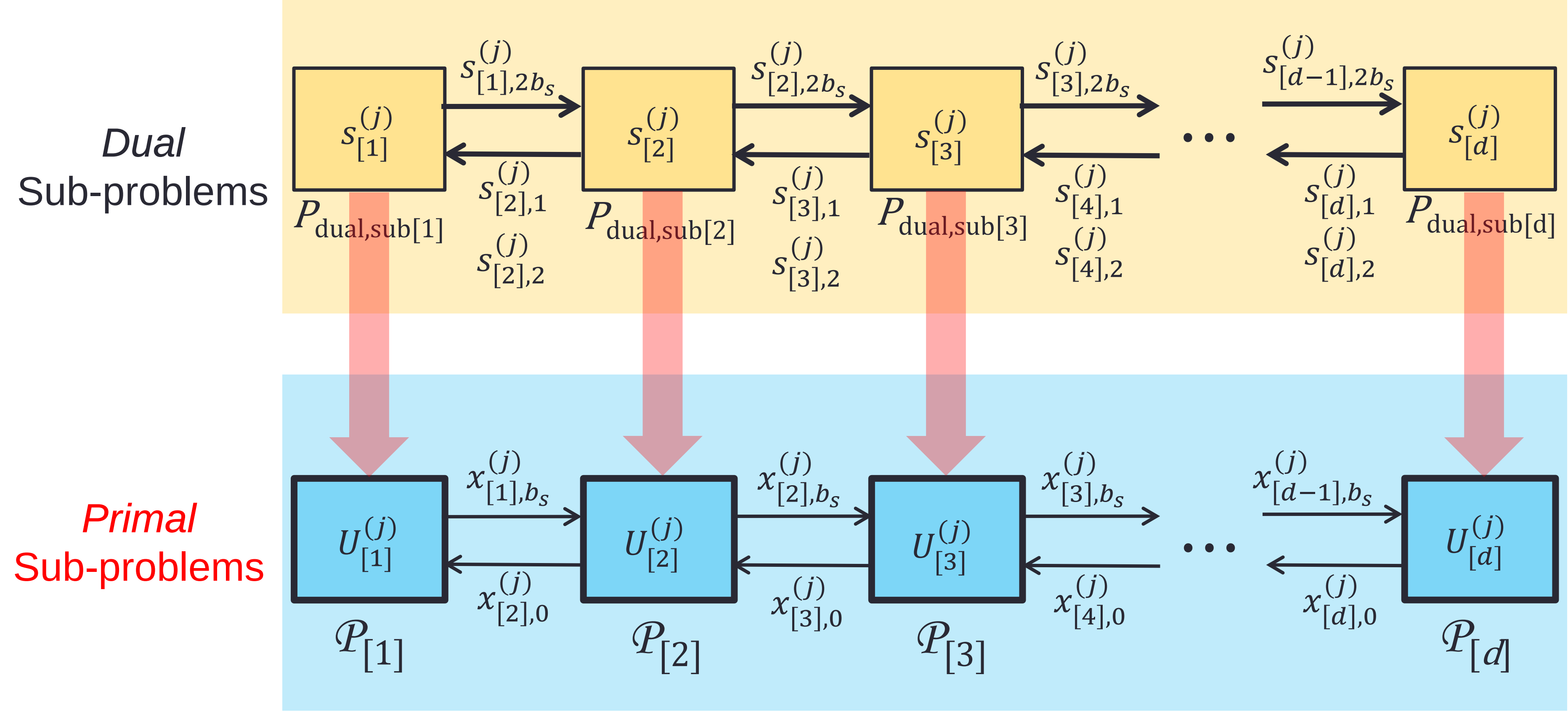
ii) The state sequence **satisfies the constraint over** $[0, N_p/d]$.

$$s^{(1)}, s^{(2)}, s^{(3)}, \ldots, s^{(j)}, \ldots,$$

Proof of Main Result

Key Idea: *Find the primal sub-problems.*

(Details are found in the paper.)



Primal sub-problems are equivalent to the modified constrained finite horizon optimal control problems.

Illustrative Example

Plant (double integrator)

$$x(t+1) = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix} u(t) \qquad x(0) = \begin{bmatrix} 1.0 \\ 1.3 \end{bmatrix}$$

$$-1 \leq u(t) \leq +1 \qquad \begin{bmatrix} -0.6 \\ -0.6 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} +2 \\ +2 \end{bmatrix}$$

MPC design parameters

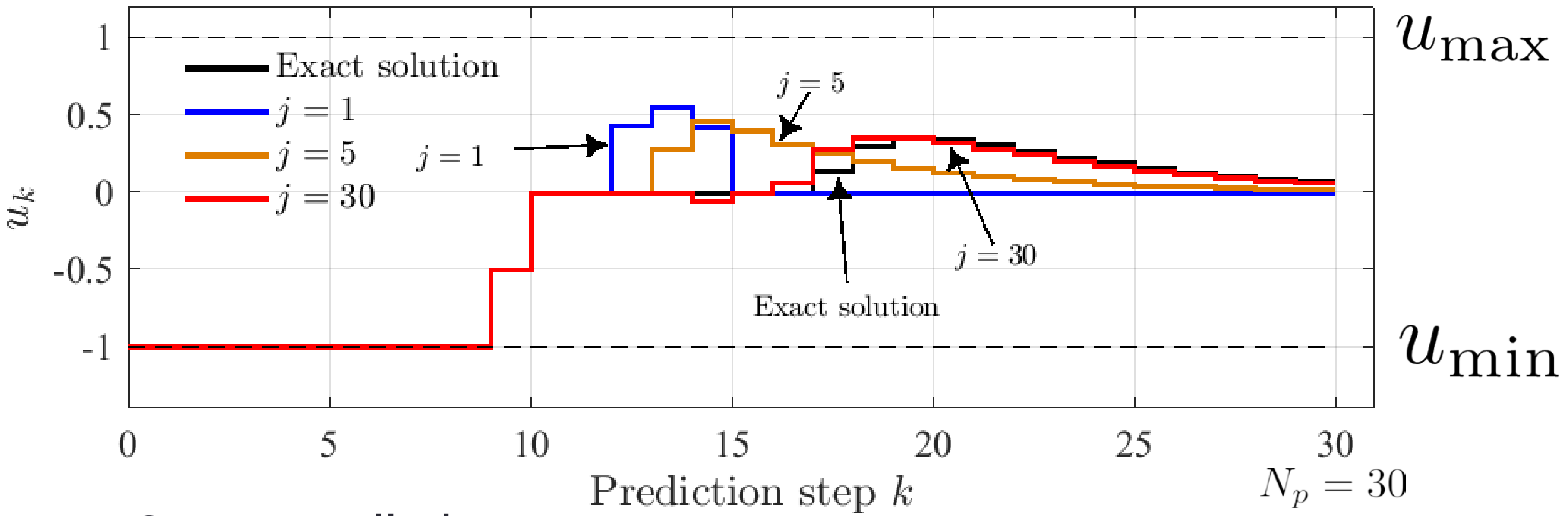
$$N_p = 30$$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 0.1 \quad Q_f$: stabilizing solution of the Riccati equation

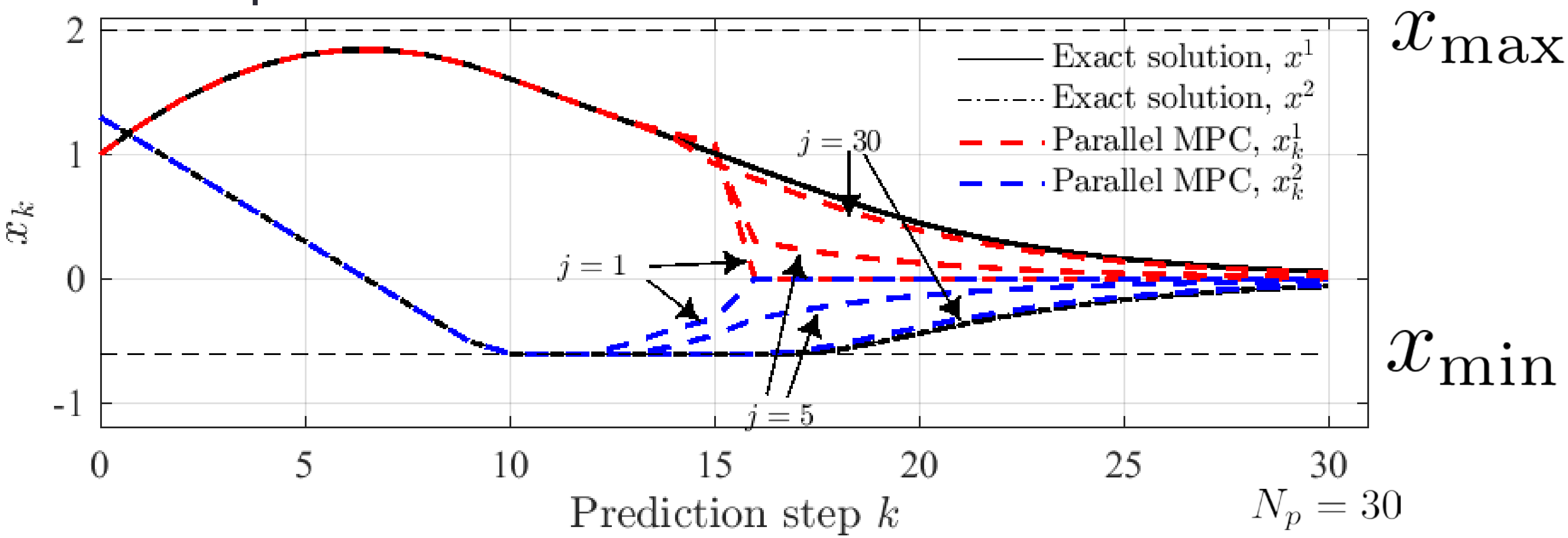
$$s_{[i]}^{(0)} = 0 \ (i = 1, 2, \ldots, d)$$

For the parallelization degree, *d=2,* we investigate the predictions obtained from the intermediate solutions.

Input prediction



State prediction



Summary & Future Work

- A method for decomposing the QP problem into several parallelizable sub-problems
- The proposed method is an iterative-based method. For any intermediate solutions:
  - Input sequence satisfies over entire horizon: state seq. satisfies over the first portion of the horizon.
  - State prediction and actual state response are not the same except for the first portion. (not addressed in this talk)
- MPC: Warm starting can be used for an initial estimate: A few iteration give a good solution. The first control move is used.

Future work

a. Actual parallel implementation and evaluate the communication overhead of parallelization
b. Convergence property should be investigated
c. Some more investigation on the constraint satisfaction (for the state)

Thank you!