Parallel Model Predictive Control for Input Constrained Linear Systems

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Introduction

- Model Predictive Control (MPC)
  - Uses real-time optimization, requires a greater computational effort
  - This can be a drawback when applied to fast systems or available computational capabilities are not sufficient.

- Two approaches to overcome the drawback
  - Explicit MPC: replace on-line optimization with an explicit function

    A. Bemporad, et al., *Automatica*, 2002

    ➢ For relatively small systems
  - Implicit MPC: speed up on-line optimization

    e.g. Y. Wang & S. Boyd, *IEEE Contr. Syst. Tech.*, 2010

    ➢ For large systems

Introduction

- Parallel computation capabilities are becoming common
  - Advent of many-core CPUs and GPUs
  - Parallelizable algorithms should be designed.
- Use of parallel computation in MPC
  - Parallel move-blocking

    S. Long, E.C. Kerrigan, K.V. Ling, and G.A. Constantinides, *CDC & ECC*, 2011
  - Tailored parallel optimization algorithms

    S.D. Cairano, M. Brand, and S.A. Bortoff, *Int. J Control*, 2013,
    I. Nielsen and D. Axehill, *CDC* 2015,
    L. Ferranti and T. Keviczky, *CDC* 2015,
    N. Hara, Y. Kagitani, and K. Konishi, *ICCAS,* 2014
- N. Hara, et al. (2014) uses the matrix splitting
  - An optimization problem is decomposed to several sub-problems.
  - Drawback: an excessive number of iterations is required.

Introduction

- In this paper,
  - Introduce a reordering of dual optimization variables for our method (N. Hara & K. Konishi, 2014).
  - Smaller sub-problems are obtained, and these can be solved in parallel.
  - The reordering is effective in improving the number of iterations.

MPC formulation

- Discrete-time LTI system

$$x(t+1) = Ax(t) + Bu(t), \ t = 0, 1, 2, \ldots,$$

$$Lu(t) \leq \bar{u},$$

$x(t) \in \mathbb{R}^n$:state $\quad u(t) \in \mathbb{R}^m$:input

$L \in \mathbb{R}^{r \times m}$ $\quad \bar{u} \in \mathbb{R}^r$:matrices for representing the input constraint

$$\min_{U \in \mathbb{R}^{(n+m)N_p}} \sum_{k=0}^{N_p-1} \{x_k^T Q x_k + u_k^T R u_k\} + x_{N_p}^T Q_f x_{N_p}$$

$$\text{s.t.} \quad x_{k+1} = Ax_k + Bu_k, \ x_0 = x(t)$$

$$Lu_k \leq \bar{u}, \ k = 0, \ldots, N_p - 1$$

$$U := \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \vdots \\ u_{N_p-1} \\ x_{N_p} \end{bmatrix}$$

$$Q, R, Q_f > 0$$

MPC formulation

- Formulated as a QP-problem

$$\min_{U} \ \frac{1}{2}U^T H_{PP} U$$
$$\text{s.t.} \ \ VU \leq W$$
$$V_{eq}U = W_{eq}x_0, \ \ x_0 = x(t),$$

$H_{PP}, V, W, V_{eq}, W_{eq}$

appropriate matrices

- MPC
  - We need to solve the QP-problem in real-time.
  - This motivates us to reduce the computational time.

Parallel formulation of MPC

- Dual of the QP problem

$$\min_{z} \ \frac{1}{2} z^T H_{DP} z + q_{DP}^T z$$

$$\text{s.t. } z_a \geq 0_{rN_p}$$

$$z := \begin{bmatrix} z_a \\ z_b \end{bmatrix} \in \mathbb{R}^{(r+n)N_p}$$

$$H_{DP} := \begin{bmatrix} V \\ V_{eq} \end{bmatrix} H_{PP}^{-1} \begin{bmatrix} V \\ V_{eq} \end{bmatrix}^T,$$

$$q_{DP} := \begin{bmatrix} W \\ W_{eq} x_0 \end{bmatrix} \in \mathbb{R}^{(r+n)N_p}$$

The optimal solution of the dual problem

$$z^* = \begin{bmatrix} z_a^* \\ z_b^* \end{bmatrix}$$

→

The optimal solution of the original QP problem

$$U^* = -H_{PP}^{-1}(V^T z_a^* + V_{eq}^T z_b^*)$$

Parallel formulation of MPC

Definition: Matrix splitting

H :symmetric matrix

(M, K) is a splitting of H if $H = M + K$.

A splitting (M, K) is a regular splitting if $M - K > 0$.

$$\min_{z} \ \frac{1}{2} z^T H_{DP} z + q_{DP}^T z$$
$$\text{s.t.} \ z_a \geq 0_{rN_p}$$

→ $H_{DP} = M + K$

regular splitting

Theorem 1: (E. Yamakawa & M. Fukushima, 1997; Z.Q. Luo & P. Tseng, 1992)

$z^{(j)} (j = 0, 1, 2, \dots)$ converges to the optimal solution.

$$z^{(j+1)} := \arg \min_{z \in \mathbb{R}^{(r+n)N_p}} \frac{1}{2} z^T M z + (K z^{(j)} + q_{DP})^T z$$
$$\text{s.t.} \ z_a \geq 0_{rN_p}$$

Parallel formulation of MPC

- Main idea of parallelization based on Theorem 1 lies in choosing a block diagonal matrix M in the splitting.

$$H_{DP} = M + K$$

QP-problem

$$\min_{z \in \mathbb{R}^{(r+n)N_p}} \frac{1}{2} z^T M z + (K z^{(j)} + q_{DP})^T z$$

$$\text{s.t. } z_a \geq 0_{rN_p}$$

↓

Independent sub-problems

- If H_{DP} has a structure close to a block diagonal matrix, the above problem approximates the dual problem effectively.
  - However, actual H_{DP} has not: excessive number of iterations (N. Hara & K. Konishi, 2014)

Parallel formulation of MPC

- Key idea: introduce a reordering of the optimization variables.

$$P_\pi = \left[\begin{array}{cccc|cccc} I_r & & & & 0 & & & \\ 0 & & & & I_n & & & \\ & I_r & & & & 0 & & \\ & 0 & & & & I_n & & \\ & & \ddots & & & & \ddots & \\ & & & I_r & & & & 0 \\ & & & 0 & & & & I_n \end{array}\right]$$

$$z = P_\pi^T s, \quad s \in \mathbb{R}^{(r+n)N_p}$$

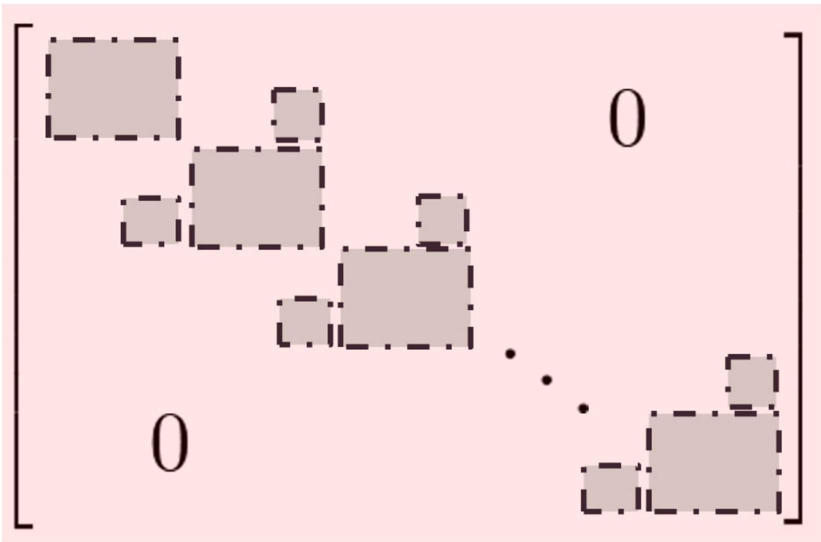
$$\frac{1}{2} z^T H_{DP} z + q_{DP}^T z = \frac{1}{2} s^T \underbrace{P_\pi H_{DP} P_\pi^T}_{H} s + \underbrace{q_{DP}^T P_\pi^T}_{q^T} s$$

Parallel formulation of MPC

$$\min_{s} \ \frac{1}{2} s^T H s + q^T s \quad \text{s.t. } \bar{s} \geq 0_{rN_p}.$$

$$\bar{s} := \begin{bmatrix} s_1 \\ s_3 \\ \vdots \\ s_{2N_p-3} \\ s_{2N_p-1} \end{bmatrix}$$

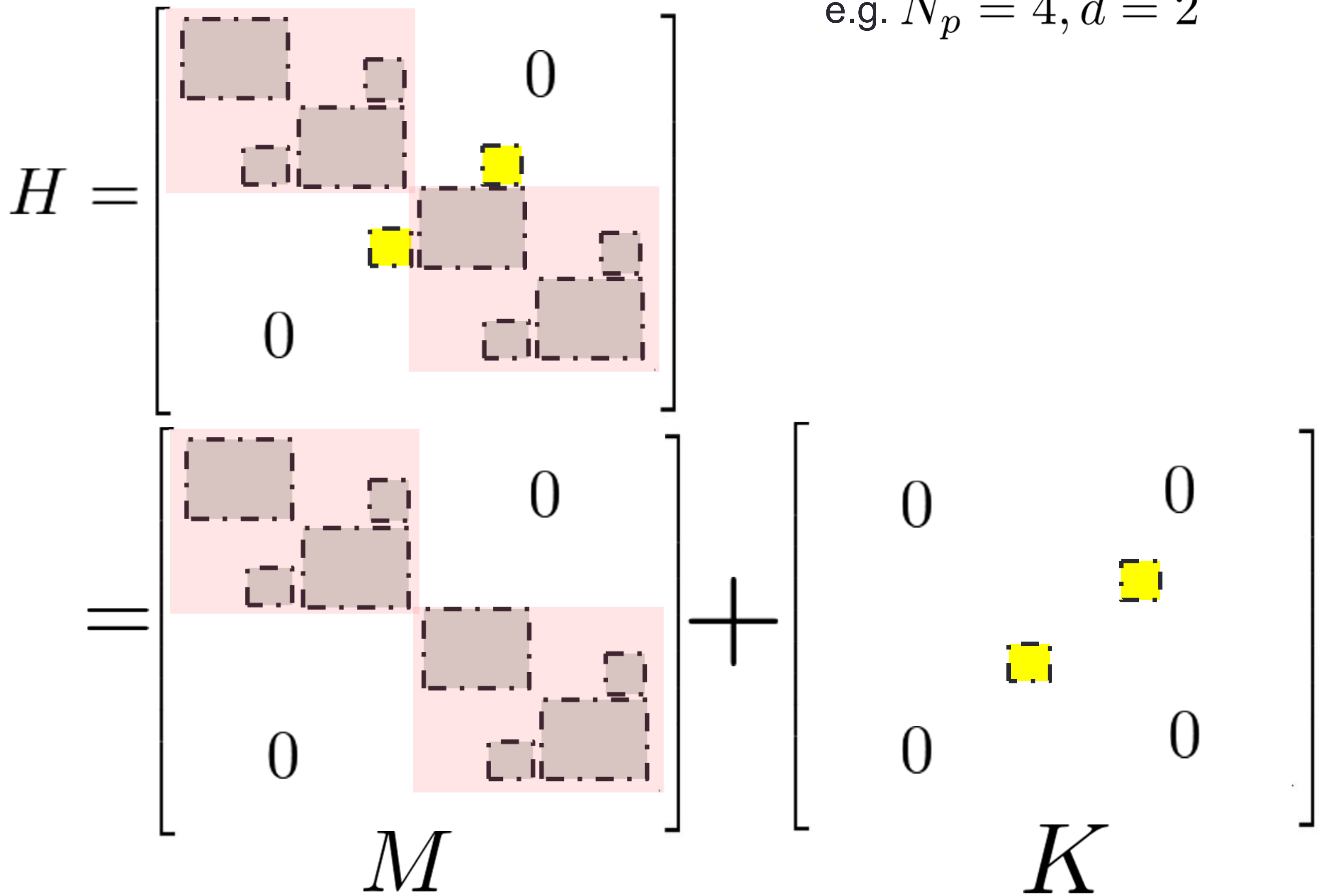
$H =$



$$q^T = \begin{bmatrix} \bar{u} \\ -Ax_0 \\ \bar{u} \\ 0_n \\ \bar{u} \\ 0_n \\ \vdots \\ \bar{u} \\ 0_n \end{bmatrix}^T$$

1. Choose parallelization degree: $d \geq 1$ (devisor of N_p)
2. Extract a block diagonal matrix and build a splitting of H

e.g. $N_p = 4, d = 2$



3. Obtain a regular splitting

$$(M_{\epsilon+}, K_{\epsilon-})$$

$$M_{\epsilon+} := M + \epsilon I_{(r+n)N_p} \quad K_{\epsilon-} := K - \epsilon I_{(r+n)N_p}$$

$$\epsilon : \epsilon > \frac{\epsilon_0}{2}, \;\; \epsilon_0 := |\min(\lambda_{\min}(M-K), 0)|,$$

4. Iterative equation involving d sub-QP problems:

$$\theta_i^{(j+1)} := \arg \min_{\theta_i \in \mathbb{R}^{(r+n)b_s}} \frac{1}{2}\theta_i^T (M_{\epsilon+})_i \theta_i + ((K_{\epsilon-})_i \theta^{(j)} + q_i)^T \theta_i$$

$$\text{s.t.} \quad \bar{\theta}_i \geq 0_{rb_s}$$

$$i = 1, 2, \ldots, d, \quad j = 0, 1, 2, \ldots,$$

✓The problem size reduced to $1/d$

✓The regular splitting ensures the convergence.

Parallel formulation of MPC

- The negative effective of the approximation is small when
  1. $K_{\epsilon-}$ is relatively small
  2. The 'guess solution' $\theta^{(j)}$ is close to the optimal solution

Lemma 2:

$$\|K_{\epsilon-}\|_F^2 = \frac{d-1}{2}\|AQ^{-1}\|_F^2 + (r+n)N_p\epsilon^2$$

$\|\cdot\|_F$:Frobenius norm

- A quantitative measure for evaluating the effective of splitting

$$\frac{\|K_{\epsilon-}\|_F}{\|H\|_F}$$

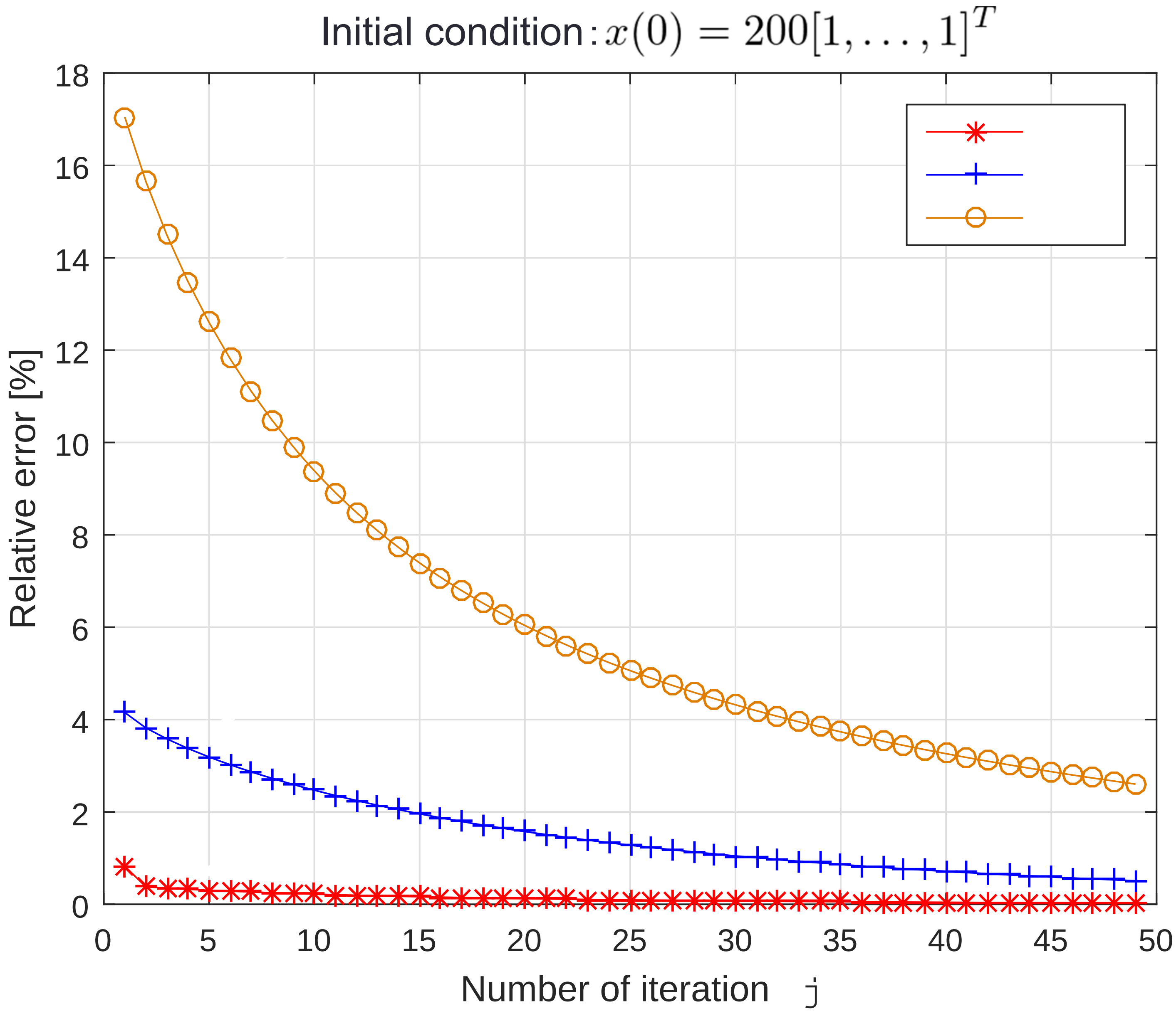
Illustrative Examples

- Plant
  - Distillation column model created by mkoilfr.m (J.M. Maciejowski, Chapter 9, *Predictive Control with Constraints*, 2002)
  - 22 states, 3 inputs
  - Input constraint: $-0.1 \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} \le u(t) \le +0.1 \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$
- Design parameters

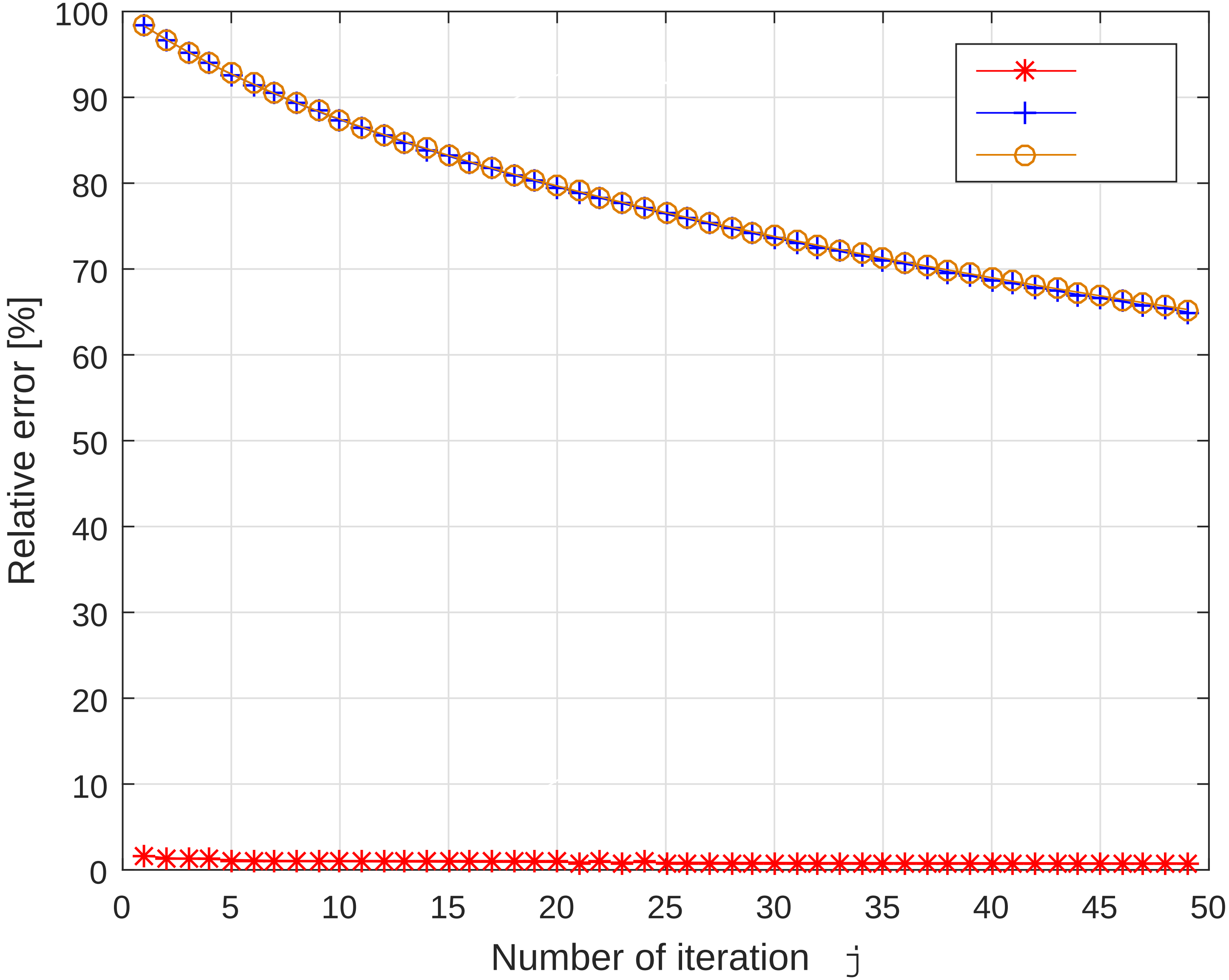
  $N_p = 100,\ Q = I_{20}, R = 0.1 I_3$

  Q_f :solution of the Riccati equation
- Optimization
  - quadprog with default algorithm (interior-point-convex) and parameters
  - Sub-problems are solved sequencially

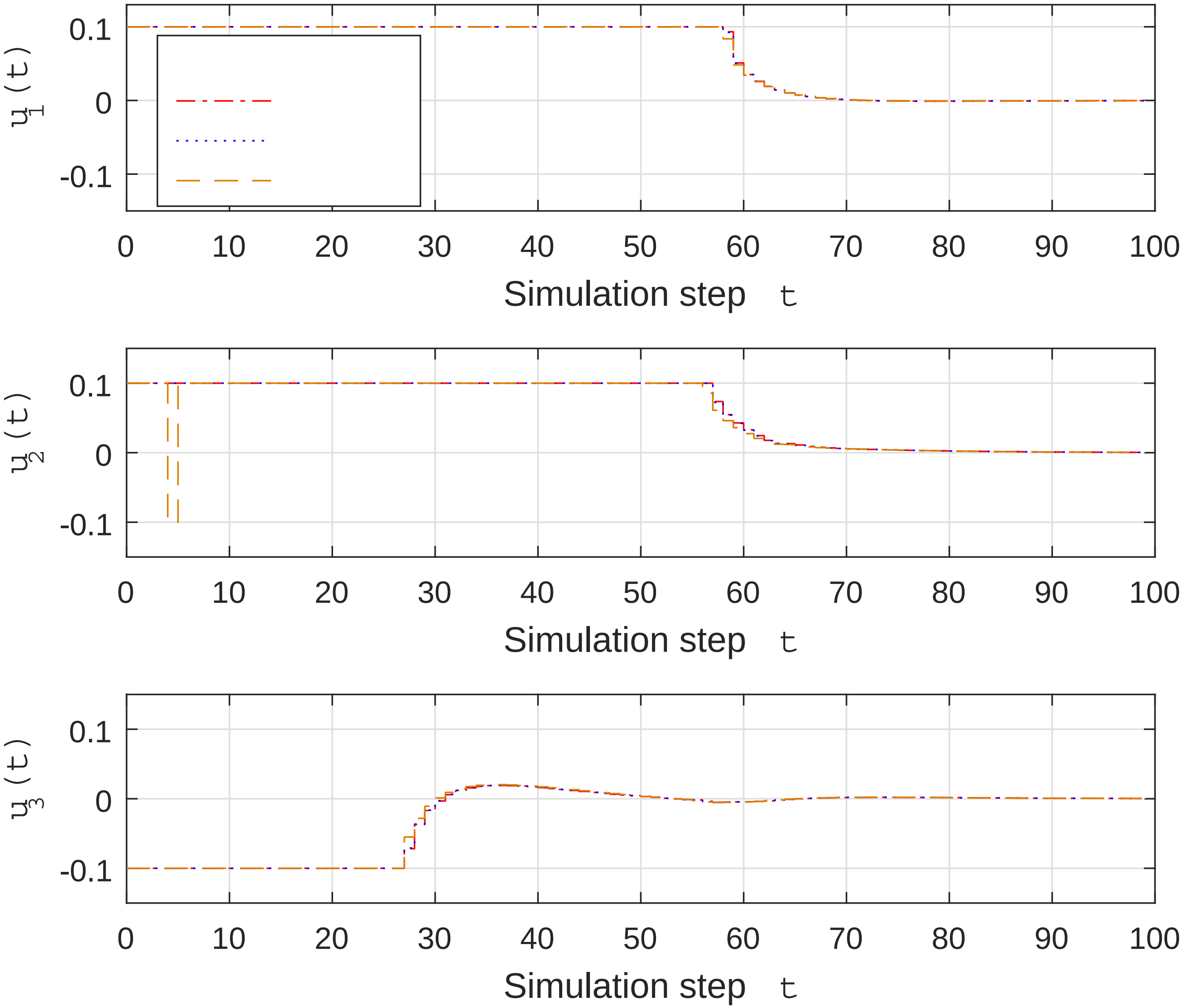
Convergence Property



Convergence Property (previous method)

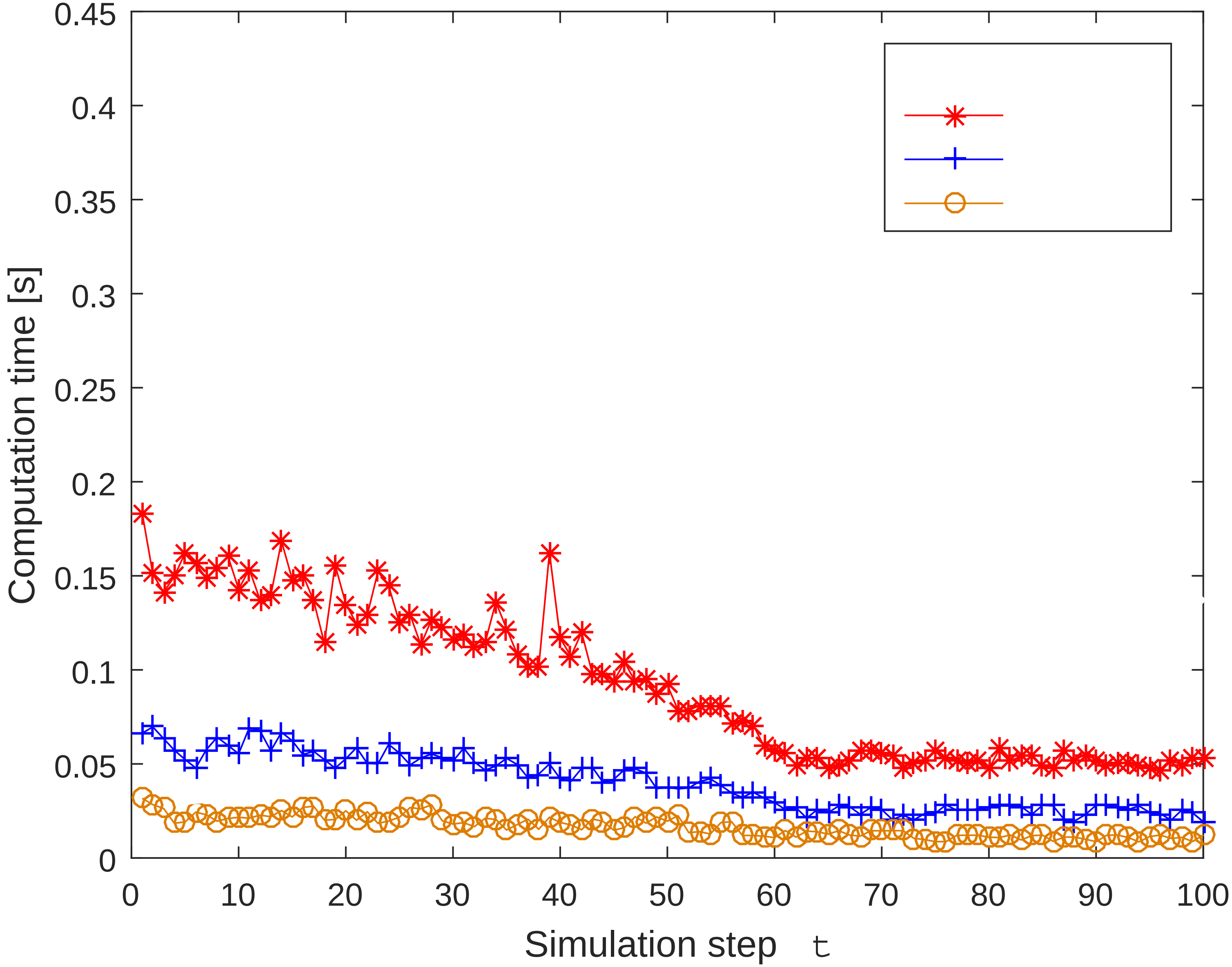


Closed-loop response (control inputs)



Only one iteration (without warm starting)

Maximum computation time at each time step



Concluding Remarks

- Parallel MPC using the matrix splitting
  - A reordering of the dual optimization variables is effective
  - One iteration gives sufficiently accurate control inputs
  - Warm starting improves the convergence significantly.

- Future research topics
  a. Actual parallel implementation and evaluate the communication overhead of parallelization
  b. State constraint: possible but need some modification
  c. Guarantee for constraint satisfaction and stability in use of intermediate solutions: tightening constraint?

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